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Terminal Singularities and Matter in F-Theory

Master Thesis in Physics
submitted by

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2016

Abstract

In F-theory, the strong-coupling limit of type IIB string theory, geometric properties of elliptic fibrations give rise to the massless spectrum of its corresponding low-energy effective theory. The typical approach in the literature is to assume that there exists a Calabi-Yau resolution of the singular elliptic fibration. Via F/M-duality the massless spectrum can be understood in terms of M2-brane wrappings of rational curves in the resolved fiber. However this approach complicates whenever a Calabi-Yau resolution does not exist since a non-Calabi-Yau resolution breaks supersymmetry in the dual M-theory. In this thesis it is shown that terminal, i.e. non-Calabi-Yau resolvable, singularities are caused by uncharged localised matter. We establish the connection between the number of localised matter multiplets at a terminal singularity and its Milnor number which is an integer invariant of a singularity.

Zusammenfassung

In F-Theorie, dem Limes von Typ-IIB-Stringtheorie mit nicht-pertubativer String-Kopplung, sind geometrische Eigenschaften von elliptischen Faserungen verantwortlich für das masselose Spektrum der effektiven Theorie. Typischerweise wurde in der Literatur bisher angenommen, dass eine Auflösung der singulären elliptischen Faserung existiert, die selbst Calabi-Yau ist. Mit Hilfe von F/M-Dualität kann das masselose Spektrum im Bild von M2-Branen, die rationale Kurven in der aufgelösten Faser umhüllen, verstanden werden. Diese Interpretation verkompliziert sich jedoch stark, sobald die elliptische Faserung terminale Singularitäten, also Singularitäten, die nicht aufgelöst werden können ohne Supersymmetrie zu brechen, besitzt. In dieser Arbeit wird gezeigt, dass diese Singularitäten von ungeladener lokalisierter Materie verursacht wird. Wir entwickeln die Beziehung zwischen der Anzahl von Materie-Multiplets, die an einer terminalen Singularität lokalisiert sind, und der Milnorzahl, einer Invarianten von Singularitäten.

This Master Thesis has been carried out by Philipp Arras at the
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Contents

1. Introduction

The processes of the physical world as we know it from experiments happen at energies far below the Planck scale. Thus they are well described by effective field theory. One essential property of effective theories is that they are highly unconstrained and their parameters typically take arbitrary values. There is a continuous infinity of those theories which makes the whole situation fairly unsatisfactory. This is one of the reasons why it is worthwhile to study string theory. It is hoped that only a tiny and perhaps discrete fraction of the set of all possible effective field theories is compatible with string theory [Vaf05]. In other words not all effective field theories can be completed in the UV by a string vacuum. At an intuitive level the reason for this is that one has to solve the equations of motion in the chosen compactification space which puts constraints from the topological side on the theory.

In this sense it is clear that a major part of studying string theory is to analyse topological and geometrical properties of the compactification space. In this thesis we would like to analyse so-called terminal singularities of F-theory compactification spaces and give them a physical meaning. The starting point will be type IIB string theory. Some of its most important ingredients are extended objects called Dp-branes, p being an odd integer, which are (p + 1)-dimensional subspaces of ten-dimensional spacetime on which open strings end. Since stacks of Dp-branes host gauge theories, they contribute essential parts to the matter spectrum. Besides, type IIB string theory possesses two scalar fields: the Ramond field C_0 called axion and the field ϕ called dilaton, which determines the string coupling. These two real scalar fields can be combined into one complex scalar field τ called axio-dilaton.

The complex scalar τ is intimately related to D7-branes: for a D7-brane the supergravity solution of τ in ten dimensions includes a logarithm which introduces a branch cut in the two normal directions to the brane. The induced monodromy $SL(2,\mathbb{Z})$ acts on τ in the fundamental representation, i.e. like a Möbius transformation. Generally speaking, multivalued fields like τ do not allow for a proper interpretation. Fortunately, type IIB string theory itself has an $SL(2,\mathbb{Z})$ symmetry which acts on τ in the exact same way. Thus the problem of multivaluedness is cured. However, a formulation of the full string theory being intrinsically $SL(2,\mathbb{Z})$-invariant does not immediately exist.

This changed when Cumrun Vafa discovered F-theory [Vaf96] [MV96a] [MV96b]. He interpreted the complex scalar τ as the complex structure modulus of an auxiliary elliptic curve which is itself $SL(2,\mathbb{Z})$ invariant. Put differently one attaches a two-torus to every point of spacetime such that the complex structure modulus of the torus point-wise coincides with the scalar field τ. This construction has been known to mathematicians for a long time: it is an *elliptic fibration*. F-theory on an elliptically fibered Calabi-Yau manifold with base B_d (B_d being a complex d-dimensional manifold) leads to low-energy physics in $10-2d$ spacetime dimensions. Since we will focus on the case of six-dimensional low-energy physics we will consider complex two-dimensional bases.

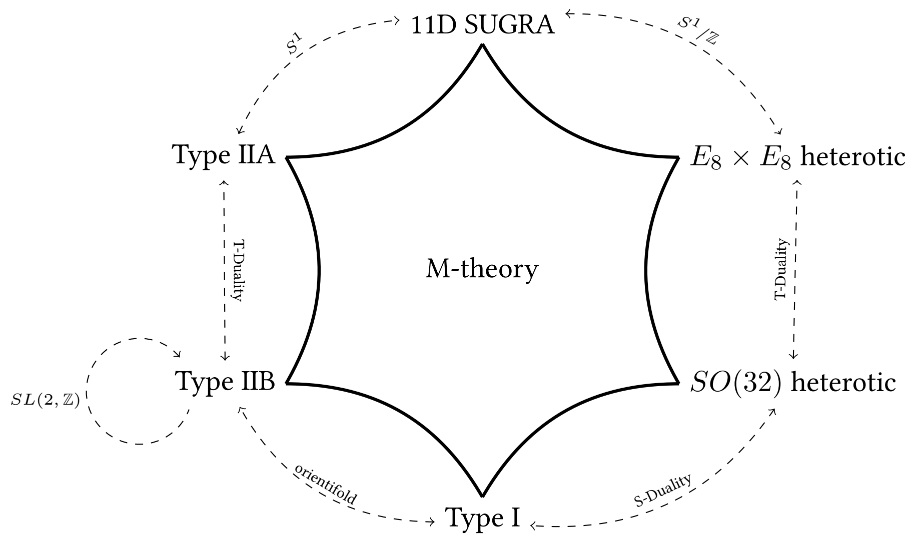


Figure 1.1.: M-theory star.

It seems that F-theory is a twelve-dimensional theory. However, the two auxiliary dimensions are very different from the other ten spacetime dimensions. This is related to the fact that no fundamental dynamical definition of F-theory exists to date. Nevertheless, it can be approached via different corners of the M-theory star (figure 1.1): it is dual to the heterotic string, to M-theory on a vanishing torus and it is the strong coupling limit of type IIB string theory.[1]

To be clear, studying F-theory on an elliptically fibered Calabi-Yau manifold Y_{d+1} means to analyse its geometric properties which give rise amongst others to the massless spectrum in $(10-2d)$-dimensional spacetime. One reason why it is worthwhile to study F-theory is that it allows for a varying axio-dilaton. However, as soon as one allows the field τ to be non-constant along B_d it is a mathematical fact that the fibration will become singular along a codimension-one locus Σ_1. By this we mean that there exists a $(d-1)$-dimensional hypersurface in B_d to which a singular elliptic curve is attached. Intuitively one can think about a singular elliptic curve as a pinched two-torus.

From the type IIB picture we know that the gauge degrees of freedom live on D-branes. In F-theory the non-abelian part of the gauge group is encoded in the singularities of the elliptic fibration at codimension one, i.e. real eight-dimensional objects called 7-branes. There is a classification of singular elliptic curves by Kodaira Kunihiko [Kod63] [Kod68] which is closely related to the classification of semi-simple Lie algebras. The gauge group on a 7-brane is the Lie group associated to the respective Lie algebra of the singular fiber [Ber+96]. To each codimension-one singularity one associates a non-abelian factor of the total gauge group in the low-energy effective action. The abelian factors of the total gauge group are associated to global properties of the fibration (the Mordell-Weil group and the Tate–Shafarevich group [BCV14]). In this thesis we will consider only gauge groups without abelian factors, i.e. fibrations with trivial Mordell-Weil group.

In the same way gauge groups and their adjoint representations are associated to codimension-one singular loci; codimension-two singularities of the fibration give rise to charged matter representa-

[1] An introduction to F-theory is given in [Wei10] and [Den08].

tions [KV97]. Codimension-two singularities are located on top of codimension-one loci, i.e. at loci where two codimension-one loci intersect. The matter living at this type of loci will be charged under the gauge group which is associated to the codimension-one locus. For completeness note that loci of higher codimension also have a physical meaning (see [BHV09a] and [BHV09b] for codimension three and [SW16] for codimension four). However, these loci are not present in our cases since we consider only complex two-dimensional bases for a reason which will become obvious in a minute.

The geometry of gauge groups is by now well-understood. In this thesis we focus on codimension-two singularities and elucidate a novel aspect of them: In addition to charged matter (as has been known) also localised, uncharged matter can be located at codimension two. In fact this uncharged matter is in one-to-one correspondence which the terminal singularities, i.e. singularities which cannot be resolved such that the resulting space is Calabi-Yau, of the elliptic fibration. What is the story behind that?

When considering elliptic fibrations with two-dimensional bases the low-energy effective theory is six-dimensional $\mathcal{N} = (1,0)$ supergravity. Its massless spectrum organises into hyper, vector and tensor multiplets (and obviously one gravity multiplet). The origin of vector and tensor multiplets in F-theory compactifications is well-understood. In other words we know which geometric properties of Y_3 give rise to a vector or a tensor multiplet in the effective theory. In this work we would like to focus on the uncharged hypermultiplets. It is known that their number is given by the number of complex structure moduli of the total space Y_3 plus one so-called universal hypermultiplet which contains the overall volume modulus and is always present.

Now the singular elliptic curves come into play. Generally, the space Y_3 will be singular. So far the literature has focused on Calabi-Yau manifolds Y_{d+1} which allow for a resolution $\tilde{Y}_{d+1}$ of the above described singularity which is itself Calabi-Yau, called *crepant resolution*, as will be discussed in more detail below. This is needed in order not to break supersymmetry in the dual M-theory along the Coulomb branch. Technically concepts like Hodge decomposition facilitate the computation of the number of complex structure moduli a lot. It can be shown that in codimension two the singularities are only resolvable if here exists only matter charged under some gauge group at these loci. As soon as uncharged matter appears the singularities won't be resolvable. This can be understood in terms of F/M-duality.

How are F-theory and M-theory related? F-theory on $\mathbb{R}^{1,5} \times Y_3$ where Y_3 is an elliptically fibered Calabi-Yau with base B_2 is dual to M-theory on $\mathbb{R}^{1,4} \times Y_3$ [Vaf96] [Wit96]. This statement means that if one compactifies the six-dimensional low-energy effective action of F-theory on a circle S^1 one obtains the effective theory of M-theory on Y_3 [IMS97] [BG12]. The radius of the compactification circle R and the volume of the fiber vol(T^2) are related to each other: $R = 1/\mathrm{vol}(T^2)$ in natural units. Thus the limit vol(T^2) $\to 0$ uncompactifies the S^1 and returns the F-theory. In this sense the auxiliary fiber of F-theory which lacks a physical interpretation a priori becomes part of the physical spacetime in the dual M-theory. The F/M-duality is crucial for the analysis of F-theory models because with its help one can extract information about the F-theory spectrum by considering the M-theory degrees of freedom and map them to F-theory via the above described circle compactification.

More concretely suppose that the elliptic fibration is singular along a codimension-one locus in the base. Then the singularity gives rise to a gauge theory living of the corresponding 7-brane which wraps the codimension-one locus and fills $\mathbb{R}^{1,5}$ as explained above. Now consider the Cartan subalgebra of this gauge group with gauge potentials $\{A_i\}$. Under the above circle compactification from six to five dimensions a vector A_i maps to a 5d-vector A_i and a 5d-scalar ξ_i and combines into 5d vector multiplets (A_i, ξ_i). Now the 5d action possess a Coulomb branch, i.e. we can give a VEV to the scalars ξ_i. On the other hand consider a resolution $\tilde{Y}_3$ of Y_3. A resolution is performed by introducing a chain of rational curves $\mathbb{P}^1_i$ in the singular fiber. Let us call the corresponding divisors by fibering $\mathbb{P}^1_i$ over the singular locus in the total space $[E_i]$. M-theory has a 3-form potential C_3 in its massless spectrum which can be reduced along the resolution divisors $[E_i]$: $C_3 = \sum_i A_i \wedge [E_i]$. In this fashion they give rise to the Cartan potentials in the 5d theory. If one identifies the scalar fields ξ_i with the Kähler moduli, i.e. the volumes, of the $\mathbb{P}^1_i$s it becomes clear that the resolution of the singularity corresponds to moving on the 5d Coulomb branch, i.e. to allow for $\langle \xi_i \rangle \neq 0$ [IMS97]. The origin of the Coulomb branch is located at $\langle \xi_i \rangle = 0$ for all i which is exactly the singular limit, i.e. the limit in which all $\mathbb{P}^1_i$ shrink to zero size.

So far we considered crepant resolutions, i.e. $\tilde{Y}_3$ is Calabi-Yau and thereby supersymmetry is unbroken. In other words this is a flat direction in the Coulomb branch. Thus all 5d states which are charged under the Cartan $U(1)$s as described above will acquire a mass. Since crepant resolutions correspond to non-trivial fiber volume of $\mathbb{P}^1_i$s the matter which is located at codimension-two and charged under the gauge group will be become massive.

What changes if one considers F-theory on a singular Calabi-Yau Y_3 which lacks a crepant resolution, i.e. possesses terminal singularities at codimension-two? The localised matter in the 6d effective theory cannot acquire a mass in a supersymmetric way any longer as one moves along a Coulomb branch in the dual M-theory. This means that the matter cannot be charged under any $U(1)$ gauge group factor in M-theory. Whenever massless matter is not charged it will remain massless when moving along any flat direction in the Coulomb branch. This indicates that there exist vanishing cycles in the fiber of the elliptic fibration which cannot be resolved without breaking supersymmetry.

All in all, this explains why a terminal singularity necessarily leads to uncharged massless matter at codimension two. In the literature a hint towards this fact was found in the special case of the so-called I_1 conifold model [BCV14] [MW15] which will be reviewed in section 6.2. The main statement of this thesis will be: *At every terminal singularity P at codimension two there are located m_P uncharged hypermultiplets which are the cause for this singularity. By m_P we denote the Milnor number of P which is a singularity characteristic.* In this fashion we develop for the first time a consistent picture of the appearance of non-crepant singularities in F-theory compactifications. In section 5.1 we will make this assertion concrete.

The reason why we consider compactifications to six dimensions is that the six-dimensional supergravity possess additionally to the usual gauge and mixed abelian anomalies, which are present in all even dimensions, gravitational anomalies and mixed non-abelian anomalies. Thus it is highly constrained by anomalies and moreover other consistency features [KT09] [KMT10a] [KMT10b] [KT11]. The gravitational anomaly which can only be cancelled if $n_H - n_V + 29n_T = 273$ where n_H, n_V and n_T are the number of hyper, vector and tensor multiplets in the massless spectrum, respectively, will be most important in our applications. The variety of anomaly sources causes severe constraints to consistent theories. Since string theory is anomaly-free, its six-dimensional low-energy effective theory has to be as well since anomalies are an IR-problem in the sense that a fundamental theory is anomaly-free if and only if its low-energy effective theory is anomaly-free. With the help of the anomaly constraints we know for sure whether a massless matter spectrum from F-theory is correct or not. Thus anomalies provide a very powerful tool to check all computations.

This thesis is organised as follows.
Chapter 2: We start out with a general introduction to anomalies and then concentrate on anomalies in six dimensions. An outline is going to provide information on how anomalies relate to topological quantities and how they can be computed with the help of index theorems. The final objective is to calculate the anomaly polynomial for six-dimensional $\mathcal{N} = (1, 0)$ supergravity.
Chapter 3: In this chapter a quick introduction to supergravity in six dimensions is given. We would like to understand how the massless matter fields organise in supersymmetry multiplets and how the anomalies arising from chiral fields and (anti-)self-dual tensor fields can be cancelled by the generalized Green-Schwarz mechanism.
Chapter 4: Now, we turn from field theory to string theory. The goal is to provide a compact introduction to F-theory which comprises all aspects which we need in the following. Additionally, we link F-theory to supergravity, i.e. reveal which geometric properties of the elliptic fibration give rise to which fields in six dimensions.
Chapter 5: After our main assertion is stated it becomes clear that in order to be able to analyse our models we need to review the calculation of the topological Euler characteristic $\chi_{\text{top}}(\tilde{Y}_3)$ of [GM00] and adopt it to our needs. Besides the concept of Milnor number is introduced and most of the notation for the following chapters is defined.
Chapter 6: We consider some prototypal models which are sensitive to the main question we would like to answer: How many hypermultiplets are located at different enhancement types, most importantly at type II $\to$ III, IV enhancements? Additionally, we supplement the analysis of [GM00] and [GM12] by explaining how the matter representations can be recovered in terms of M2-wrappings in the resolved model, i.e. reversing the F-theory limit vol$(T^2) \to 0$.
Chapter 7: Finally, we consider some models with two identical gauge group factors and present how the massless matter spectrum can be computed in these cases.

Acknowledgements

I would like to thank Daniel and Gloria for spotting typos. Thanks a lot to Ling Lin who has explained how to compute triangulations with `SageMath` and thank Antonella Grassi for sharing her knowledge about elliptically fibered Calabi-Yau threefolds and the fruitful cooperation.
Finally I would like to thank Timo Weigand for introducing me to the world of theoretical physics, in particular F-theory, and for numerous discussions in the course of last year. It was a great time for me. Besides I would particularly like to thank him for his university courses which protruded by their clarity of reasoning although they covered extremely challenging topics.

2. Anomalies

Anomalies play a crucial role in consistency considerations of both the Standard model and theories beyond the Standard model. Especially in the context of string theory and its derived low-energy effective theories anomalies are of particular importance. In our context the anomaly constraints of the low-energy effective supergravity in six dimensions of the considered F-theory models provide an efficient tool to check the correctness of the computed massless spectra.

Generally speaking the starting point when discussing anomalies is Noether's theorem a result of classical field theory: Whenever a theory is invariant under a continuous symmetry there exists a conserved current. However it can happen that the so-defined current is not conserved in the associated quantum theory in the sense that the conservation law does not hold as an operator equation. This phenomenon is called *anomaly*.

What are the implications of anomalies? A priori anomalies are not harmful to the quantum theory since the classical theory and its symmetry is only part of the QFT formalism and has no direct physical interpretation. However, if the Noether current is coupled to gauge fields, an anomaly destroys the consistency of the theory since the quantisation procedure breaks down which crucially depends on gauge symmetry (buzz word: *BRST quantisation*).

In the following we will consider two types of anomalies. First, *singlet anomalies* arise from a global symmetry whose Noether current is not coupled to any external fields. Second, the origin of *non-abelian anomalies* are local symmetries of chiral fermions. They will give rise to gauge, gravitational and mixed anomalies.

It is a well-known fact that anomalies afford a topological interpretation via index theorems [ASZ84] which reveal the connection of differential operators on a manifold to topological properties of the manifold itself. We will use this relation to compute the anomalies.

Another important fact about anomalies is Zumino's descent formalism on which all anomaly calculations in even dimensions rely: The non-abelian anomalies in $d = 2n$ are in direct correspondence to chiral anomalies in $d = 2n + 2$ [Zum83] [ZWZ84].

Finally the gravitational anomalies we would like to employ in our F-theory context were first computed by Alvarez-Gaume and Witten [AW84]. Part of this ground-breaking work is the proof that type IIB supergravity (in $d = 10$) is anomaly-free despite its chiral spectrum. We will use the results for $d = 6$. Many parts of the following chapter are based on the outline of [Avr06].

2.1. Generalities on Anomalies

For an introduction to anomalies we follow the outline of [Avr06]. As an example let us consider the theory of massless Dirac fermions transforming in some representation of a gauge group and coupled to gravity. The classical action of this theory is:

$$S = \int_{\mathbb{R}^{2n}} \sqrt{g}\bar{\psi} i \not{D} \psi.$$

This action enjoys the axial symmetry:

$$\psi \to \psi' = e^{i\alpha\Gamma_{2n+1}}\psi, \quad \bar{\psi} \to \bar{\psi}' = \bar{\psi} e^{i\alpha\Gamma_{2n+1}}. \tag{2.1}$$

Let us find the associated conserved current. To this end we follow the standard procedure, i.e. making the global parameter α position-dependent and then remove this dependence again. So if α is position-dependent, it enjoys the following commutation relation with $\not{D}$: $[\not{D}, \alpha(x)] = \Gamma^\mu \partial_\mu \alpha(x)$. Together with the usual anti-commutation relation of the Γ-matrices ($\{\Gamma^\mu, \Gamma_{2n+1}\} = 0$) we can compute the variation of the action under (2.1) with α position-dependent (assuming boundary terms vanish):

$$\begin{aligned} \delta S = S' - S &\simeq \int \bar{\psi}(1 + i\alpha\Gamma_{2n+1}) i\not{D}(1 + i\alpha\Gamma_{2n+1})\psi - \bar{\psi} i \not{D}\psi \\ &\simeq -\int \bar{\psi}(\alpha\Gamma_{2n+1}\not{D} + \not{D}\alpha\Gamma_{2n+1})\psi \\ &= -\int \underbrace{\bar{\psi}\Gamma^\mu\Gamma_{2n+1}\psi}_{=:J^\mu} \partial_\mu\alpha = \int \alpha D_\mu J^\mu. \end{aligned}$$

Removing the position dependence again and imposing $\delta S = 0$, because the transformation was a symmetry in the first place, we see that J^μ is conserved:

$$D_\mu J^\mu = 0. \tag{2.2}$$

So far we considered the classical theory. Let us now turn to its quantum version. The analogue to the classical action is the quantum effective action Γ which is defined as the logarithm of the partition function Z. We perform the calculations in Euclidean spacetime.

$$\Gamma = -\log Z = -\log \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp(-S[\psi, \bar{\psi}]). \tag{2.3}$$

Our aim is to compute the variation of Z and then combine it into the variation of Γ. The Feynman path integral integration measure $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ will transform somehow under the classical symmetry transformation. If the symmetry is non-anomalous, the measure will be invariant. Therefore, we can expand the Jacobian J of the transformation:

$$J = 1 - i\int \alpha G + \mathcal{O}(\alpha^2). \tag{2.4}$$

The so-defined quantity G is a measure for the anomaly. If it vanishes, i.e. if the Feynman measure is invariant, the theory is non-anomalous. Note that in (2.3) e^{-S} also transforms:

$$e^{-\delta S} = 1 - \int \alpha D_\mu J^\mu + \mathcal{O}(\alpha^2).$$

The above two relations can be used to calculate δZ:

$$\begin{aligned}\delta Z &= \int \mathcal{D}\psi'\mathcal{D}\bar{\psi}' e^{-S[\psi',\bar{\psi}']} - \int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{-S[\psi,\bar{\psi}]} \\ &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \left(J e^{-\delta S - S[\psi,\bar{\psi}]} - e^{-S[\psi,\bar{\psi}]} \right) \\ &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi} (J e^{-\delta S} - 1) e^{-S[\psi,\bar{\psi}]}.\end{aligned}$$

Then the variation to first order in α of the quantum effective action is given by:

$$\begin{aligned}\delta\Gamma &= -Z^{-1}\delta Z \\ &= -Z^{-1} \cdot \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \left(-i \int \alpha G - \int \alpha D_\mu J^\mu \right) e^{-S[\psi,\bar{\psi}]} \\ &= \int \alpha \left(D_\mu \langle J^\mu \rangle + iG \right).\end{aligned}$$

For α constant again, we are back at the conservation law of J^μ in operator form. However, it is now obstructed by the anomaly:

$$D_\mu \langle J^\mu \rangle = -iG.$$

At this point, we can clearly see that an anomaly, which is by definition a non-trivial transformation behaviour of the Feynman integration measure, leads to the non-conservation of the current J^μ which was conserved classically. Essentially the anomaly is encoded in the function G. The integrated version thereof ,

$$G(\alpha) := \alpha \int G = i\alpha \int D_\mu \langle J^\mu \rangle, \tag{2.5}$$

will be called *singlet anomaly*.

2.2. Calculation of the Singlet Anomaly

In this section, we would like to simplify expression (2.5) further. Since we are in Euclidean spacetime the operator $i\not{D}$ is hermitian. Therefore there exists an eigenbasis $\psi_m(x)$ with:

$$i\not{D}\psi_m(x) = \lambda_m \psi_m(x), \quad \lambda_m \in \mathbb{R},$$

which is orthonormal with respect to the inner product:

$$\langle \psi_m, \psi_n \rangle := \int \psi_m^\dagger(x)\psi_n(x) = \delta_{mn}.$$

The fermion field $\psi(x)$ can be expanded in terms of these eigenfunctions:

$$\psi(x) = \sum_n a_n \psi_n(x) \qquad \text{with } a_n = \langle \psi_m, \psi \rangle,$$
$$\bar{\psi}(x) = \sum_n \bar{\psi}_n(x)\, \bar{a}_n \qquad \text{with } \bar{a}_n = \langle \bar{\psi}, \psi_m \rangle.$$

There are two important remarks. First we choose the coefficients to be Grassmann-valued such that the basis functions are $\mathbb{C}$-valued. Second, the sum $\sum_n$ is a shorthand notation for a continuous sum.

Now we can explicitly compute how the Feynman measure transforms in order to make (2.4) explicit. It is defined as:

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = \prod_{n,m} da_n\, d\bar{a}_m.$$

If the field ψ is transformed as:

$$\psi \to \psi' = (1 + i\alpha\Gamma_{2n+1})\,\psi,$$

then the coefficients of ψ' are given by:

$$a'_n = \langle \psi_n, \psi' \rangle = \sum_m \big\langle \psi_n, (1 + i\alpha\Gamma_{2n+1})\psi_m \big\rangle\, a_m =: \sum_m C_{nm} a_m,$$
$$\bar{a}'_n = \langle \bar{\psi}', \psi_n \rangle = \sum_m \bar{a}_m \big\langle \psi_m(1 + i\alpha\Gamma_{2n+1}), \psi_n \big\rangle =: \sum_m \bar{a}_m C_{mn}.$$

Put into words the transformation matrix C which transforms a_n into a'_n is identified. Thus the integration measure transforms like:

$$\mathcal{D}\psi'\mathcal{D}\bar{\psi}' = \prod_{n,m} da'_n d\bar{a}'_m = (\det C)^{-2} \prod_{n,m} da_n d\bar{a}_m.$$

The minus sign in the exponent is due to the fact that the a_ns are Grassmann-valued. This expression can be rewritten using the well-known and very useful corollary of Jacobi's formula: $\log\det C = \operatorname{tr}\log C$.[1]

$$(\det C)^{-2} = \exp\Big(\log(\det C)^{-2}\Big) = \exp\Big(-2\log\det C\Big) = \exp\Big(-2\operatorname{Tr}\log C\Big).$$

All in all,

$$\mathcal{D}\psi'\mathcal{D}\bar{\psi}' = \underbrace{e^{-2\operatorname{Tr}\log C}}_{=J}\mathcal{D}\psi\mathcal{D}\bar{\psi}.$$

[1]We denote by Tr a trace over the functional eigenbasis of $\not{D}$ and by tr a trace over the gamma matrix indices and the group indices.

We can expand the Jacobian J to linear order in α:

$$J \simeq 1 - 2\,\mathrm{Tr}\log C = 1 - 2\,\mathrm{Tr}\underbrace{\log(1 + i\alpha\Gamma_{2n+1})}_{=i\alpha\Gamma_{2n+1}+\mathcal{O}(\alpha^2)} \simeq 1 - 2i\,\mathrm{Tr}(\alpha\Gamma_{2n+1})$$
$$= 1 - 2i\,\mathrm{tr}\sum_n \langle\psi_n, \alpha\Gamma_{2n+1}\psi_n\rangle.$$

Comparing this result to (2.4) and (2.5) we find:

$$G(\alpha) = 2\alpha\,\mathrm{Tr}\,\Gamma_{2n+1} = 2\alpha\,\mathrm{tr}\sum_n \langle\psi_n, \Gamma_{2n+1}\psi_n\rangle. \tag{2.6}$$

Note that the expression (2.6) is ill-defined. It is an infinite sum over zeros since $\mathrm{tr}\,\Gamma_{2n+1} = 0$. Therefore, we need to regulate the expression in a gauge-invariant way:

$$G(\alpha) = 2\alpha \lim_{\Lambda\to\infty} \mathrm{Tr}\left(\Gamma_{2n+1} e^{-\frac{1}{2}\left(\frac{i\not{D}}{\Lambda}\right)^2}\right). \tag{2.7}$$

2.3. Relation of Singlet Anomalies and Indices

Now we compute all possible types of singlet anomalies. This was first done in [ASZ84]. We want to follow the second approach presented in this paper. A more modern treatment of this topic is [SS04]. Our aim is to show that index theory and anomalies are intimately related.

First we consider the spin-1/2 chiral anomaly in $d = 2n$. We want to show that the anomaly can be rewritten in terms of the Dirac index of its classical equation of motion differential operator $i\not{D}$.

Let $G_{1/2}(\alpha)$ be the spin-1/2 chiral anomaly. Let us evaluate the trace in (2.7) explicitly. To this end, we note that if ψ_n is an eigenvector of $i\not{D}$ with eigenvalue λ_n, then $\Gamma_{2n+1}\psi_n$ is also an eigenvector of $i\not{D}$ but has eigenvalue $-\lambda_n$.[2] Since $i\not{D}$ is hermitian its eigenvectors with different eigenvalues are orthogonal to each other. Therefore,

$$\langle\psi_n, \Gamma_{2n+1}\psi_n\rangle = 0 \qquad \text{for all } n \neq 0.$$

In other words, the only contributions to the trace in (2.7) are the zero modes of $i\not{D}$ which we will call $\psi_0^i, i = 1, \ldots, N$. They can be split into two irreducible representations of the Clifford algebra, the positive and the negative chirality ones:

$$\Gamma_{2n+1}\psi_0^{i\pm} = \pm\psi_0^{i\pm}.$$

Then (2.7) can be rewritten in terms of the zero modes:

$$G_{1/2}(\alpha) = 2\alpha\sum_i \langle\psi_0^i, \Gamma_{2n+1}\psi_0^i\rangle = 2\alpha\left(\sum_{i_+}\langle\psi_0^{i+}, \psi_0^{i+}\rangle - \sum_{i_-}\langle\psi_0^{i-}, \psi_0^{i-}\rangle\right)$$

[2] This follows from $\{\not{D}, \Gamma_{2n+1}\} = 0$.

At this point we have to remember that we chose the eigenbasis such that the eigenvectors have unit norm. So the sum over the positive chirality zero modes corresponds to the dimension of the kernel of $i\not{D}$ acting only on positive chirality vectors. Put differently, define $i\not{D}^{\pm} := i\not{D}P^{\pm}$ with $P^{\pm}$ positive and negative chirality projectors[3] which makes the following simplification possible:

$$\begin{aligned} G_{1/2}(\alpha) &= 2\alpha\left(\dim\ker i\not{D}^{+} - \dim\ker i\not{D}^{-}\right) = 2\alpha\left(\dim\ker i\not{D}^{+} - \dim\ker(i\not{D}^{+})^{\dagger}\right) \\ &= 2\alpha\,\mathrm{ind}(i\not{D}). \end{aligned}$$

Thus we have established the relation of the spin-1/2 chiral anomaly to the Dirac index of $i\not{D}$. We are now in the powerful position to employ the Atiyah-Singer index theorem for the curved space Dirac operator [Alv83] [FW84]:

$$G_{1/2}(\alpha) = 2\alpha\,\mathrm{ind}(i\not{D}) = 2\alpha \cdot \int_{\mathcal{M}_{2n}} \left[\hat{A}(R)\,\mathrm{ch}_{\mathcal{R}}(F)\right]_{2n},$$

where $\mathcal{M}_{2n}$ is the physical spacetime. $\hat{A}$ is the A-roof genus and $\mathrm{ch}_{\mathcal{R}}(F)$ is the Chern character of F in the representation $\mathcal{R}$.

This is a remarkable result because it can be easily generalized to a spin-3/2 field and also a self-dual $(n-1)$-form which are the fields in $\mathcal{N} = (1,0)$ supergravity in six dimensions which contribute to the total anomaly (the details are presented below in chapter 3). The analogue of the Dirac operator for spin-3/2 fields is the Rarita-Schwinger operator whose index can be calculated by an index theorem as well:

$$G_{3/2}(\alpha) = 2\alpha \cdot \int_{\mathcal{M}_{2n}} \left[\hat{A}(R)(\mathrm{tr}\,e^{iR/2\pi} - 1)\,\mathrm{ch}_{\mathcal{R}}(F)\right]_{2n}.$$

Finally, consider the $(n-1)$-form anomaly. In $2n = 4k$ dimensions, there exists the potential A_{n-1} with a (anti-)self-dual field strength F_n. It gives rise to a gravitational singlet anomaly. Why is this the case? Naïvely one might expect that a bosonic field cannot contribute to the anomalies. The answer lies in the representation theory of the Lorentz group. It is a fact that the antisymmetric tensor representations of the Lorentz group with (anti-)self-dual field strengths are build as a tensor product of two Weyl representations of equal chirality. Therefore, the anomaly is related to the index of $i\not{D}_{\phi}$, the operator associated to a bispinor $\phi_{\alpha\beta}$. Its index is given by the integrated Hirzebruch polynomial $L(R)$. However, it must be corrected for three reasons: First, the second index β should be of same chirality as the first index and the potential A_{n-1} should be real. This leads to a factor $\frac{1}{4}$. Additionally, we have to multiply by (-1) because A_{n-1} obeys Bose rather than Fermi statistics. All in all, the anomaly due to the self-dual $(n-1)$-form is given by:

$$G_A(\alpha) = -\frac{\alpha}{2} \cdot \int_{\mathcal{M}_{2n}} \left[L(R)\right]_{2n}.$$

The above three results can also be obtained by an explicit field theory calculation (see [AW84]).

[3] Note that $i\not{D}^{-} = (i\not{D}^{+})^{\dagger}$.

2.4. Wess-Zumino Consistency and Descent Equations

In this section we will establish the connection of gauge anomalies in even dimensions $d = 2n$ and chiral anomalies in $d = 2n + 2$. We follow the outline of [Bil08] and [GSW85].

The crucial ingredient of this correspondence is the Wess-Zumino consistency condition whose solution is related to the characteristic classes $\mathrm{tr}\, F^{n+1}$ via the so-called descent equations. The Wess-Zumino consistency condition will constrain the form of the anomalies under infinitesimal gauge transformations in even dimensions. Recall that chiral fermions are only present in even dimensions. In odd dimensions they do not exist and therefore cannot give rise to any anomalies.

2.4.1. Wess-Zumino Consistency Condition

Let us consider an arbitrary infinitesimal gauge transformation of a gauge theory. It has the form:

$$\delta_\epsilon \Gamma[A] = \int_{\mathcal{M}_{2n}} \epsilon^\alpha(x) \mathcal{A}_\alpha(x).$$

In other words,

$$\mathcal{A}_\alpha(x) = -\left(D_\mu \frac{\delta}{\delta A_\mu(x)} \right)_\alpha \Gamma[A] =: \mathcal{G}_\alpha(x)\Gamma[A]. \tag{2.8}$$

We are interested in the commutation relation of two consecutive infinitesimal gauge transformations which can be explicitly computed:

$$[\mathcal{G}_\alpha(x), \mathcal{G}_\beta(y)] = \ldots = \delta(x-y) C_{\alpha\beta\gamma} \mathcal{G}_\gamma(x),$$

where $C_{\alpha\beta\gamma}$ are the structure constants of the Lie algebra of the gauge group. We can apply this to the quantum effective action $\Gamma[A]$ and get:

$$\mathcal{G}_\alpha(x)\mathcal{A}_\beta(y) - \mathcal{G}_\beta(y)\mathcal{A}_\alpha(x) = C_{\alpha\beta\gamma}\delta(x-y)\mathcal{A}_\gamma(x), \tag{2.9}$$

which is the *Wess-Zumino consistency condition* [WZ71].

This rather clumsy expression can be reformulated in BRST-language. Let S be the BRST-operator. Recall its defining property $SA_\mu = D_\mu c$ where $c^\alpha(x)$ is the ghost field. The action of S on any functional F local in A can be interpreted in terms of the gauge anomaly:

$$\begin{aligned} S\,F[A] &\overset{\text{chain rule}}{=} \int (D_\mu c(x))^\alpha \frac{\delta}{\delta A^\alpha_\mu(x)} F = \int c^\alpha(x) \left(-D_\mu \frac{\delta}{\delta A_\mu(x)} \right)_\alpha F \\ &\equiv \int c^\alpha(x) \mathcal{G}_\alpha(x) F. \end{aligned} \tag{2.10}$$

Clearly Γ is a local functional in A. So we can combine (2.8) and (2.10) and let S act on Γ:

$$S\,\Gamma[A] = \int d^{2n}x\, c^\alpha(x)\mathcal{A}_\alpha(x) =: \mathcal{A}[c, A].$$

The BRST-operator is nilpotent since it is a Grassmannian object. It follows that $S\mathcal{A}[c, A] = 0$. One can show by explicit computation of $S\mathcal{A}[c, A]$ that this statement is equivalent to (2.9). Thus, we have identified anomalies as BRST-closed functionals of ghost number one.
To conclude this section let us consider irrlevant anomalies, i.e. anomalies which can be cancelled by addition of a local counterterm to the action. In this sense they are only an artefact of of the chosen action which is only defined up to a total derivative. Let us show that irrelevant anomalies correspond to trivial BRST cohomology classes, i.e. $\mathcal{A}[c, A] = SF[A]$ for some local functional F. If this is the case one can add the counterterm $\Delta S = -F$ to the action which results at leading order in $\Delta\Gamma = -F$. Then $S(\Gamma + \Delta\Gamma) = S(\Gamma - F) = \mathcal{A} - SF = 0$. As we have shown above, $S\,\Gamma$ is the gauge anomaly of the theory. Thus, we have cancelled the anomaly corresponding to a BRST-trivial representative. So *relevant* anomalies are not only BRST-closed but also *not* BRST-exact. Put differently relevant anomalies are non-trivial representatives of BRST cohomology classes of ghost number one:

$$\mathcal{A}[c, A] \simeq \mathcal{A}[c, A] + SF[A] \qquad \text{for a local functional } F. \tag{2.11}$$

This is a bit unsatisfactory since we have not found a unique mathematical object characterising a gauge anomaly yet. To construct such an invariant object will be our task for the next section.

2.4.2. Derivation of Descent Equations

The descent formalism heavily depends on the language of characteristic classes of bundles and Chern-Simons forms. We introduce both concepts only at an intuitive level. For more details the reader is referred to standard maths literature. An introduction for physicists can be found in [Nak03].

Characteristic Classes

Let us consider a principle bundle on a topological space X. A *characteristic class* is a prescription to associate a cohomology class of X to the bundle. This class measures the non-triviality of the bundle, i.e. how much it differs from a simple global product. Since we do not need to dive deep into the maths of principle bundles, let us follow the simplified definition of characteristic classes of [Bil08].
A characteristic class P is a local form on a compact manifold $\mathcal{M}$ constructed from the curvature 2-form R or the field strength F of a gauge potential (gauge group G) such that its integral over $\mathcal{M}$ is sensitive to non-trivial topology. Moreover let us define $P_m(F) := \operatorname{tr} F^m = \operatorname{tr} F \wedge \ldots \wedge F$ (m times).
It is a non-trivial fact that the P_ms form a complete basis of any gauge-invariant polynomial P in F, i.e. $P(g^{-1}Fg) = P(F)$ for all $g \in G$. Furthermore it can be easily shown that the P_m are actually topological invariants.
The reason we consider the P_ms in the first place is the following property which naturally leads to the definition of Chern-Simons forms.

Claim: P_m is closed for all m.

In the convention with anti-hermitian gauge group generators the non-abelian field strength is defined as $F := dA + A^2$. Therefore, the exterior derivative is $dF = dA \wedge A - A \wedge dA = dA \wedge A + A^3 - A \wedge dA - A^3 = F \wedge A - A \wedge F$. For matrix-valued differential forms the cyclicity property of the trace generalizes to $\operatorname{tr} \omega^{(q)} \omega^{(p)} = (-1)^{pq} \operatorname{tr} \omega^{(q)} \omega^{(p)}$ for $\omega^{(p)}, \omega^{(q)}$ being a p,q-form, respectively. Thus, $dP_m = d \operatorname{tr} F^m = m \operatorname{tr} dF \wedge F^{m-1} = m \operatorname{tr}(F \wedge A - A \wedge F) \wedge F^{m-1} = 0$.
#

Chern-Simons Forms

Since the P_m are closed they are locally exact:

$$P_m(F) = dQ_{2m-1}(F, A) \quad \text{locally.}$$

The $(2m-1)$-forms $Q_{2m-1}(F, A)$ are called *Chern-Simons forms*. There are three important things to note. First, Chern-Simons forms are only defined up to an exact piece. Second, they are defined only for odd degree in A. Finally and perhaps most striking, Chern-Simons forms are not gauge invariant. All three properties will play a crucial role in the following discussion. The two lowest-rank Chern-Simons forms are:

$$\begin{aligned} Q_3 &= \operatorname{tr}\Big(A \wedge F - \tfrac{1}{3} A^3\Big), \\ Q_5 &= \operatorname{tr}\Big(A \wedge F^2 - \tfrac{1}{2} A^3 F + \tfrac{1}{10} A^5\Big). \end{aligned}$$

Just for fun let us check whether the expression for Q_3 is correct. Use $\operatorname{tr} dF = \operatorname{tr}(dA \wedge A + A \wedge dA)$:

$$\begin{aligned} dQ_3 &= \operatorname{tr}\Big(dA \wedge F - A \wedge dF - dA \wedge A^2\Big) \\ &= \operatorname{tr}\Big(dA \wedge dA + \cancel{dA \wedge A^2} - A \wedge dA \wedge A + A^2 \wedge dA - \cancel{dA \wedge A^2}\Big) \\ &= \operatorname{tr}\Big(dA \wedge dA + 2\, dA \wedge A^2\Big) = \operatorname{tr} F^2 \ \checkmark \end{aligned}$$

Descent Equations

As we have already said Chern-Simons forms are not gauge-invariant. It seems that they are completely useless for any analysis of gauge theories. However appearances are deceptive. Actually, Chern-Simons forms are central in the context of anomalies of gauge theories. Their crucial property is that the gauge-variation of a Chern-Simons form is closed,

$$d\, \delta_v Q_{2m-1}(F, A) = \delta_v\, dQ_{2m-1}(F, A) = \delta_v P_m(F) = 0,$$

and therefore again locally exact:

$$\delta_v Q_{2m-1}(F, A) = dQ^{(1)}_{2m-2}(v, F, A) \quad \text{locally,}$$

where $dQ^{(1)}_{2m-2}$ is defined only up to an exact piece. Additionally, recall that Q_{2m-1} was defined only up to an exact piece, too: $Q_{2m-1} \simeq Q_{2m-1} + d\alpha_{2m-2}$. Thus, $\delta Q_{2m-1} \simeq \delta Q_{2m-1} + \delta\, d\alpha_{2m-2} = \delta Q_{2m-1} + d\,\delta\alpha_{2m-2}$. Therefore, $Q^{(1)}_{2m-2} \simeq Q^{(1)}_{2m-2} + \delta\alpha_{2m-2}$. Altogether, $dQ^{(1)}_{2m-2}$ is defined up to an exact and a gauge piece.
To make the above explicit let us continue our two examples and calculate $Q^{(1)}_2$ and $Q^{(1)}_4$.

Claim: $\delta_v \mathrm{tr} A^l \wedge F^k = \mathrm{tr}\left[(dv \wedge A^{l-1} + A \wedge dv \wedge A^{l-2} + \ldots + A^{l-1} \wedge dv) \wedge F^k\right]$.

We proof the claim by induction. In our conventions $\delta_v A = dv + [A, v]$ and $\mathrm{tr} F^k$ is gauge invariant.

- Initial step ($l = 1$): $\delta_v\, \mathrm{tr} A \wedge F^k = \mathrm{tr}\Big(dv \wedge F^k + [A, v] \wedge F^k\Big) = \mathrm{tr}\Big(dv \wedge F^k\Big)$. ✓
- Induction step ($l \to l + 1$): $\delta_v \mathrm{tr}\, A^{l+1} F^k = \mathrm{tr}\,\big(\delta_v A \wedge A^l \wedge F^k + A \wedge \delta_v (A^l \wedge F^k)\big)$. Plugging in the induction hypothesis completes the proof.

#

The above claim helps us to compute the two examples:

- $\delta_v Q_3 = \delta_v \mathrm{tr}\Big(A \wedge F - \frac{1}{3}A^3\Big) = \mathrm{tr}\Big(dv \wedge F - \frac{1}{3}(dv \wedge A^2 + A \wedge dv \wedge A + A^2 \wedge dv)\Big)$. The expression simplifies to $\delta_v Q_3 = \mathrm{tr}(dv \wedge dA)$. Thus, $Q^{(1)}_2 = \mathrm{tr}(v \wedge dA)$ up to exact terms.
- Similarly, $\delta_v Q_5 = \mathrm{tr}\,\left[dv \wedge (dA \wedge dA + \frac{1}{2} dA^3)\right]$ and $Q^{(1)}_4 = \mathrm{tr}\, v \wedge d\Big(A \wedge dA + \frac{1}{2}A^3\Big)$ plus exact terms.

Recap

For future reference we condense the essential results.

- The properties of $P_m := \mathrm{tr}\, F^m$ are:

$$dP_m = 0 = \delta_v P_m. \tag{2.12}$$

- The descent equations hold only locally and are given by:

$$P_m = dQ_{2m-1}, \quad \delta_v Q_{2m-1} = dQ^{(1)}_{2m-2}. \tag{2.13}$$

- The objects Q_{2m-1} and $Q^{(1)}_{2m-2}$ are defined up to:

$$Q_{2m-1} \simeq Q_{2m-1} + d\alpha_{2m-2}, \tag{2.14}$$

$$Q^{(1)}_{2m-2} \simeq Q^{(1)}_{2m-2} + \delta\alpha_{2m-2} + d\beta_{2m-3}. \tag{2.15}$$

2.4.3. Relation of Descent Equations and Gauge Anomalies

After having introduced gauge anomalies and the descent formalism we can understand the interplay of both. We will show how gauge anomalies (2.8) and the descent equations (2.13) are related. The

anomaly will be proportional to $Q^{(1)}_{2m}$. The ambiguity of defining the anomaly will correspond to ambiguities (2.14) and (2.15). Via the descent equations we will find the term P_{m+1} which is not ambiguous. So we will characterise the gauge anomaly uniquely by the $(2m+2)$-form P_{m+1}.
It is a general QFT fact that anomalies must involve the Levi-Civita symbol $\epsilon^{\mu_1 \cdots \mu_{2m}}$ and thus can be interpreted as a $2m$-form $Q^{(1)}_{2m}(c, A)$ which is of ghost number 1:

$$\mathcal{A}[c, A] \sim \int_{\mathcal{M}_{2m}} Q^{(1)}_{2m}(c, A).$$

Recall the Wess-Zumino condition (2.11): $S\mathcal{A} = 0$ and $\mathcal{A} \neq S(\ldots)$ for consistent anomalies. Since S acts like a gauge transformation on A we can rewrite the descent equations as: $P_{m+1} = dQ_{2m+1}$ and $SQ_{2m+1} = dQ^{(1)}_{2m}$ (locally). We now claim that the so defined $Q^{(1)}_{2m}$ is a solution to the Wess-Zumino condition.

Claim: $\int_{\mathcal{M}_{2m}} Q^{(1)}_{2m}(c, A)$ is a solution to the Wess-Zumino condition

There are two things to show. First, we have to show that the expression is BRST-closed and then we have to show that it is not BRST-exact.

1. $SQ_{2m+1} = dQ^{(1)}_{2m}$. So $0 = S(SQ_{2m+1}) = S\,dQ^{(1)}_{2m} = -d(SQ^{(1)}_{2m})$ because S is nilpotent and d and S anti-commute. Since the auxiliary $(2m+2)$-dimensional space may have non-trivial topology it is not clear that closed forms are exact which is what we need to finish the proof. Because $Q^{(1)}_{2m}$ is of ghost number one and therefore linear in c, $SQ^{(1)}_{2m}$ will be bilinear in the ghost field and can be assumed to be globally defined. In other words the topology of the space can be chosen such that all closed $2m$-forms are exact. Then, $SQ^{(1)}_{2m} = d\alpha^{(1)}_{2m-1}$ and $S\int_{\mathcal{M}_{2m}} Q^{(1)}_{2m} = 0$. For more details the reader is referred to [Bil08].

2. Suppose that $Q^{(1)}_{2m}$ is exact; $Q^{(1)}_{2m} = S\alpha_{2m}$ with α_{2m} a function of A only. Then $SQ_{2m+1} = dQ^{(1)}_{2m} = dS\alpha_{2m} = -Sd\alpha_{2m}$. All in all, $S(Q_{2m+1} + d\alpha_{2m}) = 0$. There does not exist any gauge invariant forms of odd degree. Thus, already $Q_{2m+1} + d\alpha_{2m}$ must vanish. This is the contradiction: Acting with d on the expression reveals $dQ_{2m+1} = 0$ but $dQ_{2m+1} = P_m \neq 0$. Thus the assumption is false and $Q^{(1)}_{2m}$ is not BRST-exact. Finally, we have to show that the integrated version is not BRST-exact, too. To this end suppose $\int_{\mathcal{M}_{2m}} Q^{(1)}_{2m} = S\int_{\mathcal{M}_{2m}} \beta_{2m} = \int_{\mathcal{M}_{2m}} S\beta_{2m}$. The integrands of both sides of the equation can only differ by an exact term: $Q^{(1)}_{2m} + d\gamma^{(1)}_{2m-1} = S\beta_{2m}$. But $Q^{(1)}_{2m}$ has been defined only up to an exact term in the first place. We can redefine it and absorb $d\gamma^{(1)}_{2m-1}$. It follows that $Q^{(1)}_{2m} = S\beta_{2m}$ which we have excluded already.

#

We can conclude that in even dimensions the gauge anomaly is given by:

$$\mathcal{A}[c, A] = C\int_{\mathcal{M}_{2m}} Q^{(1)}_{2m}(c, A) + S(\ldots), \tag{2.16}$$

where C is a proportionality constant. It corresponds to CP_{m+1} via the descent formalism. If $C = 0$

the anomaly is not relevant and can be compensated by local counterterms in the action.

2.5. The Anomaly Polynomial

So far we have outlined the relation of non-abelian anomalies in $2n$ dimensions and chiral anomalies in $2n+2$ dimensions through the descent equations and we have already computed the singlet anomaly. Thus we are now able to actually compute the anomalies we are interested in. We follow [Avr06] again. First rotate back to Minkowski spacetime (anomaly $G_M = -G_E$ and quantum effective action $\Gamma_M = i\Gamma_E$). Then the anomalies are given by:

$$G^{(2n+1)}(\alpha) = -\frac{1}{\pi}\alpha \int_{\mathcal{M}_{2n+2}} \hat{I}_{2n+2},$$

with

$$\begin{aligned}
\hat{I}^{1/2}_{2n+2} &= 2\pi \left[\hat{A}(R)\,\mathrm{ch}_{\mathcal{R}}(F)\right]_{2n+2},\\
\hat{I}^{3/2}_{2n+2} &= 2\pi \left[\hat{A}(R)(\mathrm{tr}\, e^{iR/2\pi} - 1)\,\mathrm{ch}_{\mathcal{R}}(F)\right]_{2n+2},\\
\hat{I}^{A}_{2n+2} &= 2\pi \left[\frac{1}{2}\cdot\frac{1}{4}\cdot L(R)\right]_{2n+2}.
\end{aligned}$$

Recalling the descent equations we obtain the Minkowski anomalies in $2n$ dimensions:

$$d\hat{I}_{2n+1} = \hat{I}_{2n+2}, \quad \delta_{v,\lambda}\hat{I}_{2n+1} = -d\hat{I}^1_{2n}.$$

Thus,

$$G^{(2n)}(v,\lambda) = \int_{\mathcal{M}_{2n}} \hat{I}^1_{2n}.$$

We are now in the position to actually compute the anomalies. In this thesis we are especially interested in the anomalies of six-dimensional supergravity. For this reason we focus on the case $n = 3$ from now on. It will turn out that a convenient redefinition is:

$$I_8 := -16\,(2\pi)^3\,\hat{I}_8.$$

We now need the explicit formulae for the A-roof genus, the Chern character and the Hirzebruch polynomial:

$$\begin{aligned}
\hat{A}(R) &= 1 + \frac{1}{(4\pi)^2}\frac{1}{12}\,\mathrm{tr}R^2 + \frac{1}{(4\pi)^4}\left(\frac{1}{360}\,\mathrm{tr}R^4 + \frac{1}{288}(\mathrm{tr}R^2)^2\right) + \cdots,\\
\mathrm{ch}_{\mathcal{R}}(F) &= 1 - \frac{1}{(2\pi)^2}\frac{1}{2}\mathrm{tr}F^2 + \frac{1}{(2\pi)^4}\frac{1}{24}\mathrm{tr}F^4 - \cdots,\\
L(R) &= 1 - \frac{1}{(2\pi)^2}\frac{1}{6}\mathrm{tr}R^2 + \frac{1}{(2\pi)^4}\left(-\frac{7}{180}\mathrm{tr}R^4 + \frac{1}{72}(\mathrm{tr}R^2)^2\right) + \cdots.
\end{aligned}$$

We split our redefined I_8 into the terms depending on F, R and both:

$$\begin{aligned} I_8^{1/2}(F) &= \tfrac{2}{3}\mathrm{tr}F^4, & I_8^{3/2}(F) &= \tfrac{10}{3}\mathrm{tr}F^4, \\ I_8^{1/2}(R) &= \tfrac{1}{360}\mathrm{tr}R^4 + \tfrac{1}{288}(\mathrm{tr}R^2)^2, & I_8^{3/2}(R) &= \tfrac{49}{72}\mathrm{tr}R^4 - \tfrac{43}{288}(\mathrm{tr}R^2)^2, \\ I_8^{1/2}(F,R) &= -\tfrac{1}{6}\mathrm{tr}R^2\,\mathrm{tr}F^2, & I_8^{3/2}(F,R) &= \tfrac{19}{6}\mathrm{tr}R^2\,\mathrm{tr}F^2, \\ \left(I_8^{1/2}(F_X,F_Y) = 4\mathrm{tr}F_X^2\,\mathrm{tr}F_Y^2\right), & & I_8^{A}(R) &= -\tfrac{7}{90}\mathrm{tr}R^4 + \tfrac{1}{36}(\mathrm{tr}R^2)^2. \end{aligned} \tag{2.17}$$

The expression in parenthesis is only present if the spin-1/2 field is charged under more than one simple factor of the total gauge group.

This is our final result. Now we can write down the anomaly of a six-dimensional theory with an arbitrary spectrum by adding up the above contributions accordingly.

2.6. Anomaly Cancellation

In the last section we have given the explicit formulae for anomalies in arbitrary (even) dimensions. They are encoded in the anomaly polynomial which is a $(d+2)$-form. In section 2.4 we have shown that the anomaly polynomial is a unique representative of the anomaly whereas the actual expression for the anomaly is not uniquely determined (see (2.15)). It was always possible to add a total derivative $d\beta_{d-1}$ or a pure gauge term $\delta\alpha_d$ to the action. This reveals the two ways in which potential anomalies arising through a chiral spectrum can be cancelled:

1. Vanishing of the anomaly polynomial. In the case of ten-dimensional type IIB supergravity it is also called *"miraculous" anomaly cancellation.*

2. Vanishing of $Q_d^{(1)}$ by adding a counterterm to the action, i.e. showing that the anomaly is not a relevant anomaly. In the case of ten-dimensional type I supergravity coupled to Yang-Mills theory it is referred to as anomaly cancellation via the *Green-Schwarz mechanism.*

The first one is called "miraculous" because type IIB supergravity has a chiral spectrum (a self-dual 5-form, two real chiral gravitinos and two anti-chiral Majorana dilatinos). Therefore it was believed to be anomalous for a long time. But if one carefully computes the contributions to the anomaly polynomial I_{12}, one finds that it identically vanishes. From the field theoretic point of view this insight came as a big surprise. However the string theory community must have known that the hasty conclusion that type IIB supergravity is anomalous had to be wrong because it is the field theoretical limit of type IIB string theory which is definitely anomaly-free.

In our theories the anomalies will not cancel miraculously. Therefore, we focus on the Green-Schwarz mechanism and follow the lines of [GS84].

Let us consider ten-dimensional $\mathcal{N}=1$ supergravity coupled to a gauge group G. The gauge and gravitational anomalies in ten dimensions can be described by the gauge invariant anomaly polynomial I_{12} which can be computed with the help of index theorems. Let us introduce some notation. Analogously to the Yang-Mills field strength and potential $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = dA + A\wedge A$ (with anti-hermitian generators) we introduce the curvature 2-form for gravity:

$$R = \frac{1}{2} R_{\mu\nu} dx^\mu \wedge dx^\nu = d\omega + \omega \wedge \omega,$$

where ω is a 10×10 antisymmetric matrix in the fundamental representation of $SO(1,9)$.

The anomalies of the theory come from various sources. First, the supergravity multiplet contains a left-handed Majorana-Weyl gravitino and a right-handed Majorana-Weyl spinor. Both are not charged under the gauge group. Therefore we have to include only the R and not the F dependence in the anomaly polynomial. Additionally, there are $n = \dim G$ copies of a left-handed Majorana-Weyl spinor. By supersymmetry they transform in the same representation as the gauge fields, the adjoint representation. Thus we have to take traces over the adjoint representation. To compute the anomaly we have to add up all contributions to the anomaly polynomial according to the above outline [Bil08]:[4]

$$64(2\pi^5)\, I_{12} = 64(2\pi^5) \Big(\underbrace{I_{12}^{3/2}(R) - I_{12}^{1/2}(R)}_{\text{SUGRA sector}} + \underbrace{I_{12}^{1/2}(F,R)}_{\text{SYM sector}} \Big) \tag{2.18}$$

$$= \left(\frac{\dim G - 496}{5670}\, \mathrm{tr} R^6 + \frac{\dim G + 224}{4320}\, \mathrm{tr} R^4\, \mathrm{tr} R^2 + \frac{\dim G - 64}{10368}\, (\mathrm{tr} R^2)^3 \right) \\ -(\mathrm{tr} F^2) \left(\tfrac{1}{180}\, \mathrm{tr} R^4 + \tfrac{1}{144}\, (\mathrm{tr} R^2)^2 \right) + \tfrac{1}{576}\, \mathrm{tr} F^4 \mathrm{tr} R^2 - \tfrac{1}{360}\, \mathrm{tr} F^6. \tag{2.19}$$

It seems that the theory is anomalous. However, we will escape through a loophole: *If* I_{12} factorises into $I_{12} = (\mathrm{tr} R^2 + k\, \mathrm{tr} F^2) \wedge I_8$ where I_8 is a closed gauge-invariant 8-form depending on F and R and k is a real number, *then* there exists a local counterterm $\Delta\Gamma$ which cancels the anomaly.

Let us show this assertion. Let $B = B_{\mu\nu} dx^\mu \wedge dx^\nu$ be the 2-form potential form of the $\mathcal{N} = 1$ supergravity multiplet. It turns out that one has to define its field strength H in the following way in order to be gauge invariant and consistent with supersymmetry:

$$H = dB - \tilde{k}\, Q_3(A,F) + k\, Q_3(\omega, R),$$

with $k, \tilde{k}$ being numbers which depend on the normalization of B and H and the conventions for the trace in Q_3. Let us rescale and set $\tilde{k} = 1$. Because H is gauge-invariant and $\delta Q_3 = Q_2^{(1)}$:

$$\delta_{v,\lambda} B = Q_2^{(1)}(v, A, F) - k Q_2^{(1)}(\lambda, \omega, R).$$

Define $\Delta\Gamma = \int B \wedge X_8$ where X_8 is closed and gauge-invariant. This is the counterterm which we add to the action in order to cancel the anomalies of the above form. So let us show that this $\Delta\Gamma$ leads to the contribution $\Delta I_{12} = (\mathrm{tr}\, R^2 + k\, \mathrm{tr}\, F^2) \wedge I_8$ in the anomaly polynomial. The induced anomaly

[4] Note that the trace over powers of F is in the adjoint representation whereas the traces of the curvature form are taken in the fundamental representation.

$\mathcal{A}$ is given by:

$$\begin{aligned}
\mathcal{A} = \delta_{v,\lambda}(\Delta\Gamma) &= \int \delta_{v,\lambda}(B) \wedge X_8 \quad \text{since } X_8 \text{ is gauge invariant.} \\
&= \int \Big(Q_2^{(1)}(v, A, F) - k\, Q_2^{(1)}(\lambda, \omega, R) \Big) \wedge X_8 \\
&= -\int \Big(\delta Q_3(A, F) - k\, \delta Q_3(\omega, R) \Big) \wedge X_7 \quad \text{since } X_8 = dX_7 \text{ and } dQ_2^{(1)} = \delta Q_3. \\
&=: \int I_{10}^{(1)}.
\end{aligned}$$

Now, using the descent formalism we ascent from $I_{10}^{(1)}$ to I_{12}:

$$\begin{aligned}
dI_{10}^{(1)} &= \Big(\delta Q_3(A, F) - k\, \delta Q_3(\omega, R) \Big) \wedge X_8 \quad \text{since } \delta Q_3 \text{ is closed.} \\
&\stackrel{!}{=} \delta_{v,\lambda} I_{11}
\end{aligned}$$

Because X_8 is gauge-invariant I_{11} is given by:

$$I_{11} = \Big(Q_3(A, F) - k\, Q_3(\omega, R) \Big) \wedge X_8.$$

Finally, the contribution to the anomaly polynomial is:

$$I_{12} = dI_{11} = \Big(Q_4(F) - k\, Q_4(R) \Big) \wedge X_8.$$

By definition $Q_4(F) = \mathrm{tr}\, F^2$ and $Q_4(R) = \mathrm{tr}\, R^2$, which shows the assertion: Whenever the anomaly polynomial factorises into $I_{12} = (\mathrm{tr}\, R^2 + k\, \mathrm{tr}\, F^2) \wedge I_8$ the anomaly is not relevant and can be cancelled by adding a local counterterm to the action.

The next step is to show under which conditions the anomaly polynomial factorises. Of course the $\mathrm{tr}\, F^6$ and the $\mathrm{tr}\, R^6$ terms have to vanish in order to satisfy the factorisation condition. The latter fixes $\dim G = 496$.

At first glance the $\mathrm{tr}\, F^6$ won't vanish. However, there exist certain gauge groups in which $\mathrm{tr}\, F^6$ can be expressed as a linear combination of $\mathrm{tr}\, F^4\, \mathrm{tr}\, F^2$ and $(\mathrm{tr}\, F^2)^3$. To cut a long story short there are exactly two Lie groups without abelian factors which meet the conditions: $SO(32)$ and $E_8 \times E_8$. Theories with abelian factors suffer from other inconsistencies. As a side remark, the results form this section establish *string universality* in ten dimensions: Every consistent ten-dimensional supergravity is the low-energy limit of a string theory.

The main concern of this thesis is six-dimensional supergravity where a similar mechanism is at work to cancel the anomalies. However the details are much more complicated. The intuitive reason for this is that there are more than one self-dual tensor fields in six-dimensional supergravity opposed to only one self-dual tensor in ten-dimensional supergravity. Although this complicates the calculation a lot, the general picture remains the same. The details were worked out by [Sag92].

3. $\mathcal{N} = (1, 0)$ Supergravity in Six Dimensions

In this chapter we would like to give an overview over the most important features of $\mathcal{N} = (1, 0)$ supergravity in six dimensions for our purposes. As a rough overview: The fields organise into one gravity multiplet and an arbitrary number of tensor, vector and hypermultiplets. All multiplets contribute to the anomaly since all contain chiral fields or (anti-)self-dual tensors. Finally it is described how the generalized Green-Schwarz mechanism is able to cancel the anomalies in certain cases which are constrained by a number of equations.

3.1. Multiplet Structure

By $\mathcal{N} = (1, 0)$ in six dimensions we mean supersymmetry generated by eight real supercharges. The massless states form representations of the little group in six dimensions, $SO(4)$, which is isomorphic to $SU(2) \times SU(2)$. Restricting to spins less or equal two the different types of multiplets are summarized in table 3.1 [NS97]. Certainly, there is one gravity multiplet. The number of tensor, vector and hypermultiplets are variable and will be denoted by n_T, n_V and n_H, respectively.
Having a look at the multiplet structure one immediately notices two things. First the spectrum is chiral and therefore potentially the source of anomalies. Second it comprises (anti-)self-dual 2-forms. Both properties will be crucial in the following discussion.
The appearance of (anti-)self-dual 2-forms is conspicuous. They obstruct a naïve Lagrangian formulation of the theory since the standard kinetic term will identically vanish by the duality condition.[1] However, there exists a so-called *pseudo-action.* It is an action which leads to the correct classical equations of motion *after* imposing the duality constraints. In other words, the duality constraints are imposed on the equations of motion and not directly on the action. Therefore it is called pseudo-action. In order to write down its bosonic part we need some notation.

- **2-forms.** The 2-form from the gravity multiplet and the tensor 2-forms are combined into $B^\alpha, \alpha = 1, \ldots, n_T + 1$. The scalars from the tensormultiplet are viewed as coordinates j^α on the scalar manifold $\mathcal{M}_{\text{scalar}} = {}^{SO(1,n_T)}/_{SO(n_T)}$. We can introduce a metric $\Omega_{\alpha\beta}$ with mostly minus Lorentzian signature $(1, n_T)$ subject to the constraint $\Omega_{\alpha\beta} j^\alpha j^\beta = 1$. Additionally we can define another metric on $\mathcal{M}_{\text{scalar}}$: $g_{\alpha\beta} := 2j_\alpha j_\beta - \Omega_{\alpha\beta}$ where we raise and lower indices with $\Omega_{\alpha\beta}$: $j_\alpha := \Omega_{\alpha\beta} j^\beta$. $g_{\alpha\beta}$ is positive definite. Finally let G^α be the field strength of the 2-form potential B^α.

[1] Actually, it is possible to write down a Lagrangian in the case $n_T = 1$. Here the self-dual 2-form in the gravity multiplet can be combined with the anti-self-dual 2-form of the tensormultiplet which gives a 2-form without duality constraints. This was worked out in [NS97].

One Gravity multiplet $(g_{\mu\nu}, \psi^-_\mu, B^+_{\mu\nu})$	One graviton	$(\mathbf{1},\mathbf{1})$
	One left-handed Weyl gravitino	$2\,(\frac{\mathbf{1}}{\mathbf{2}},\mathbf{1})$
	One self-dual 2-form	$(\mathbf{1},\mathbf{0})$
n_T tensormultiplets $(B^-_{\mu\nu}, \chi^+, \phi)$	One anti self-dual 2-form	$(\mathbf{0},\mathbf{1})$
	One right-handed Weyl tensorino	$2\,(\mathbf{0},\frac{\mathbf{1}}{\mathbf{2}})$
	One real scalar	$(\mathbf{0},\mathbf{0})$
n_V vectormultiplets (A_μ, λ^-)	One vector	$(\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}})$
	One left-handed Weyl gaugino	$2\,(\frac{\mathbf{1}}{\mathbf{2}},\mathbf{0})$
n_H hypermultiplets $(\psi^+, 4\varphi)$	One right-handed Weyl hyperino	$2\,(\mathbf{0},\frac{\mathbf{1}}{\mathbf{2}})$
	Four real scalars	$4\,(\mathbf{0},\mathbf{0})$

Table 3.1.: The multiplets of $\mathcal{N} = (1,0)$ supergravity in six dimensions. In the last column the $SU(2)\times SU(2)$ representation is displayed.

- **Vectors.** For brevity let us consider a gauge group G with only one simple factor. Let $\mathfrak{g}$ be the Lie algebra of G and A be the $\mathfrak{g}$-valued gauge 1-form. Then $F := dA + A\wedge A$ is the associated non-abelian field strength. We can define the Chern-Simons form $\omega_{CS} := \mathrm{tr}\,(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)$ where tr is the trace in the respective representation of $\mathfrak{g}$. The two basic properties of the Chern-Simons form are (see section 2.4.2):

$$\delta\omega_{CS} = \mathrm{tr}\, d\lambda\wedge dA, \quad d\omega_{CS} = \mathrm{tr}\, F\wedge F.$$

- **Hyper Scalars.** There are four real scalars in each hypermultiplet: $q^U, U = 1,\dots,4n_H$. They can be viewed as real coordinates of a quaterionic manifold with metric h_{UV}. The covariant derivative is $Dq^U := dq^U + A^I(T^R_I q)^U$ where I runs over the generators of the gauge group and T^R_I are the generators acting on q^U in some representation R. As the details are not important for our purposes the reader is referred to [FS90] and [And+97].

- **Gravity.** For the graviton we employ the vielbein formalism. ω is the $\mathfrak{so}(1,5)$-valued spin connection 1-form. It is determined by the torsion-free condition $de + \omega\wedge e = 0$. Then $\delta w = dl + [\omega, l]$ where l is a $\mathfrak{so}(1,5)$-valued 0-form. The field strength is $R := d\omega + \omega\wedge\omega$, the curvature 2-form. Analogously we can define the gravitational Chern-Simons form $\omega^{CS}_{grav} := \mathrm{tr}(\omega\wedge d\omega + \frac{2}{3}\omega\wedge\omega\wedge\omega)$ with the transformation law $\delta\omega^{CS}_{grav} = \mathrm{tr}\, dl\wedge d\omega$ and the exterior derivative $d\omega^{CS}_{grav} = \mathrm{tr} R\wedge R$. Let $\mathcal{R}$ be the curvature scalar.

Then the pseudo-action is given by:

$$S = \int_{\mathcal{M}_6} \frac{1}{2}\mathcal{R} * 1 - h_{UV} Dq^U \wedge *Dq^V - \frac{1}{4} g_{\alpha\beta} G^\alpha \wedge *G^\beta - \frac{1}{2} g_{\alpha\beta} dj^\alpha \wedge *dj^\beta$$
$$-2\,\Omega_{\alpha\beta} j^\alpha b^\beta \mathrm{tr} F\wedge *F - V * 1$$

One can summarize that a six-dimensional theory with gravity and $\mathcal{N} = (1,0)$ supersymmetry is characterised by the following data:

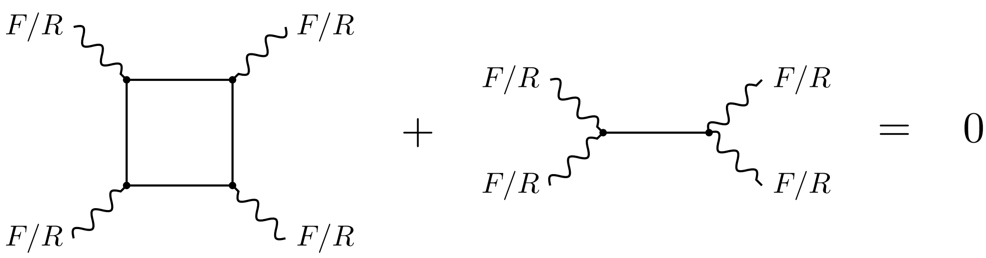


Figure 3.1.: Green-Schwarz mechanism in six dimensions. The wavy lines shall represent the fields F and R and the straight lines stand for the two-forms of the theory.

1. The number of tensormultiplets n_T.
2. The gauge group G which determines the number of vectormultiplets $n_V = \dim G$.
3. The representation R under which the hypermultiplets transform. This determines amongst others the number of charged and uncharged hypermultiplets.

Naturally a question arises: Are all possible choices of the above three parameters allowed or do specific combinations lead to inconsistent theories? The answer is tightly connected to the anomalies of the theories and the possibility to cancel them.

3.2. Anomaly Cancellation and the Generalized Green-Schwarz Mechanism

From section 2.5 we know that anomalies in six-dimensional supergravity are encoded in the 8-form I_8 from which we could compute the actual anomaly via the descent formalism. All we need to do in order to analyse the anomalies is to combine table 3.1 and expressions (2.17) with the number of hyper, vector and tensormultiplets of the theory of consideration.
Anomaly cancellation in six dimensions works similar to anomaly cancellation in ten dimensions via the Green-Schwarz mechanism (see section 2.6). However it is much more complicated since there are tensormultiplets in the theory which means that we are faced with more than one (anti-)self-dual 2-form. In 1992 Augusto Sagnotti generalized the Green-Schwarz mechanism to six dimensions [Sag92]. As in ten dimensions the anomaly polynomial I_8 has to meet a factorisation condition which implies that the anomaly can be cancelled by a local counterterm (see figure 3.1). To state the result we generalize the notation to more than one simple gauge group factor: Let us denote their respective field strengths by F_i. $\operatorname{tr} F_i$ is the trace in the adjoint representation of the gauge group factor G_i. Then the factorization condition is [Tay11]:

$$I_8 = \tfrac{1}{2}\Omega_{\alpha\beta}\, I_4^\alpha \wedge I_4^\beta \qquad \text{with } I_4^\alpha = \tfrac{1}{2}a^\alpha \operatorname{tr} R^2 + \sum_i b_i^\alpha \tfrac{2}{\lambda_i} \operatorname{tr} F_i^2, \tag{3.1}$$

where a^α, b_i^α are in $\mathbb{R}^{1,T}$ and λ_i are the normalization constants for each simple group factor G_i (e.g. $\lambda_{SU(N)} = 1, \lambda_{E_8} = 60$).

Whether the anomaly polynomial factorizes or not depends on the chosen gauge group, the number of tensormultiplets and the chosen matter representations. In this way anomaly cancellation constraints the space of consistent six-dimensional supergravities.

In the sequel we would like to translate condition (3.1) into a bunch of tangible equations which must be satisfied by a consistent spectrum. In the anomaly polynomial there are terms proportional to $\mathrm{tr}R^4, \mathrm{tr}F^4, (\mathrm{tr}R^2)^2, \mathrm{tr}F^2\,\mathrm{tr}R^2$ and so forth. Let us demonstrate by a few examples how to arrive at the actual constraints.

The $\mathrm{tr}\,R^4$ and $(\mathrm{tr}\,R^2)^2$ terms. From (2.17) we have the following contributions:

$$\begin{aligned} I_8^{1/2}(R) &= \tfrac{1}{360}\mathrm{tr}\,R^4 + \tfrac{1}{288}(\mathrm{tr}\,R^2)^2, \\ I_8^{3/2}(R) &= \tfrac{49}{72}\mathrm{tr}\,R^4 - \tfrac{43}{288}(\mathrm{tr}\,R^2)^2, \\ I_8^{A}(R) &= -\tfrac{7}{90}\mathrm{tr}\,R^4 + \tfrac{1}{36}(\mathrm{tr}\,R^2)^2. \end{aligned}$$

Recall that for fermions of opposite chirality and anti-self-dual 2-form one has to add the terms with opposite sign. Taking a look at table 3.1 we conclude:

$$\begin{aligned} I_8(R) &= -I_8^{3/2}(R) + I_8^A(R) + n_T\left[I_8^{1/2}(R) - I_8^A(R)\right] - n_V I_8^{1/2}(R) + n_H I_8^{1/2}(R) \\ &= \tfrac{1}{360}\left(n_H - n_V + 29n_T - 273\right)\mathrm{tr}R^4 + \tfrac{1}{288}\left(n_H - n_V - 7n_T + 51\right)(\mathrm{tr}R^2)^2. \end{aligned}$$

Let us analyse the two terms in turn. Comparing to (3.1) one observes that the term proportional to $\mathrm{tr}\,R^4$ has to vanish. Thus, anomaly cancellation constraints the spectrum by

$$n_H - n_V + 29\,n_T - 273 = 0, \tag{3.2}$$

which is a result we will use many times in the sequel. Obviously, the term proportional to $(\mathrm{tr}R^2)^2$ is constrained as well:

$$\tfrac{1}{2}\cdot\left(\tfrac{1}{2}\right)^2\cdot(a\cdot a) \stackrel{!}{=} \tfrac{1}{288}\left(n_H - n_V - 7n_T + 51\right)$$

Simplifying and plugging in (3.2) gives:

$$a\cdot a = 9 - n_T.$$

The $\mathrm{tr}\,F^4$ term. First note that the gravity and the tensormultiplets are not charged under the gauge group. Therefore only the gauginos and hyperinos contribute to the $\mathrm{tr}\,F^{2k}$ terms. Let n_R^i be the dimension of the representation R of gauge group G_i under which the hypermultiplets transform.

Rep.	A_R	B_R	C_R
$\square$	1	1	0
Adjoint	$2N$	$2N$	6
Antisym.	$N-2$	$N-8$	3

Table 3.2.: Values of group theoretic constants A_R, B_R and C_R for $SU(N), N \geq 4$. For $SU(2)$ and $SU(3)$, A_R is given in the table, $B_R = 0$ and C_R is computed by $C_R + B_R/2$ from the table with $N = 2, 3$. [Tay11][Erl94]

Then there are the following contributions:

$$\begin{aligned} I_8(F)|_{F^4} &= \sum_i \left[-I_8^{1/2}|_{F^4} + \sum_R n_R^i I_8^{1/2}|_{F^4} \right] \\ &= -\tfrac{2}{3} \sum_i \left[\mathrm{tr} F_i^4 - \sum_R n_R^i \mathrm{tr}_R F_i^4 \right] . \end{aligned}$$

Introducing the notation $\mathrm{tr}_R F^4 := B_R \,\mathrm{tr}\, F^4 + C_R (\mathrm{tr}\, F^2)^2$ and imposing $I_8(F)|_{F^4} = 0$ we obtain the constraint:

$$B^i_{Adj.} - \sum_R n_R^i B_R^i = 0.$$

All constraints. The other terms are analysed in a similar fashion. Let us summarize all constraints [Tay11]:

$$\begin{aligned}
\mathrm{tr} R^4 : &\qquad n_H - n_V + 29 n_T - 273 = 0, && (3.3)\\
\mathrm{tr} F^4 : &\qquad 0 = B^i_{Adj.} - \sum_R n_R^i B_R^i, && (3.4)\\
(\mathrm{tr} R^2)^2 : &\qquad a \cdot a = 9 - n_T, && (3.5)\\
\mathrm{tr} F^2 \mathrm{tr} R^2 : &\qquad a \cdot b_i = \tfrac{1}{6} \lambda_i \left(A^i_{Adj.} - \sum n_R^i A_R^i \right), && (3.6)\\
(\mathrm{tr} F^2)^2 : &\qquad b_i \cdot b_i = \tfrac{1}{3} \lambda_i^2 \left(\sum n_R^i C_R^i - C^i_{Adj.} \right), && (3.7)\\
\mathrm{tr} F_i^2 \mathrm{tr} F_j^2 : &\qquad b_i \cdot b_j = \sum_{R,S} n_{RS}^{ij} A_R^i A_S^j \quad \text{for } i \neq j. && (3.8)
\end{aligned}$$

n_{RS}^{ij} is the number of matter fields transforming in the $R \times S$ representation of $G_i \times G_j$. The values of the group theoretical constants A_R, B_R and C_R are given in table 3.2.[2]

Note that we considered semi-simple gauge groups only. If $U(1)$ factors are present it is still possible to cancel the anomalies but the whole formalism is more involved [Erl94][Hon06].

The above constraints (3.4) to (3.8) are necessary conditions which must be met in order to have a consistent theory. Our analysis in the following chapters will rely heavily on these equations.

[2]Note that there is an additional factor two in equation (3.8) in [Tay11]. By explicit computation and comparison to e.g. [Avr06] one can confirm that this factor is not present and presumably is a typo.

4. F-theory

F-theory is the non-pertubative formulation of type IIB string theory. It is able to capture strong coupling behaviours and can handle compactifications with varying axio-dilaton. F-theory can be approached from three corners of the M-theory star:

- F-theory is dual to $E_8 \times E_8$ heterotic theory.
- F-theory is strongly coupled Type IIB theory with 7-branes and varying string coupling.
- F-theory is dual to M-theory on a torus T^2 with $\mathrm{area}(T^2) \to 0$.

In this thesis we focus on the latter two. In Type IIB there are two massless real scalars: the dilaton ϕ and the axion C_0. They can be combined into a complex scalar field, the axio-dilaton:

$$\tau := ie^{-\phi} + C_0 \equiv \frac{i}{g_s} + C_0.$$

The axio-dilaton is only defined up to the $SL(2,\mathbb{Z})$ action:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad \text{with} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}).$$

Besides we note that there is a natural $SL(2,\mathbb{Z})$ action on the complex structure of a two-torus T^2. Putting those two ideas together marks the birth of F-theory in 1996 [Vaf96]. Cumrun Vafa introduced a fictitious elliptic curve (i.e. a two-torus) and identified its complex structure with the axio-dilaton. Varying τ along ten-dimensional spacetime corresponds to variation of the complex structure modulus of the auxiliary elliptic curves. This construction is a standard object in geometry: it is an elliptic fibration. In other words we attach to each point of ten-dimensional spacetime of IIB theory an elliptic curve. The non-triviality of the fibration is a measure for how strongly the axio-dilaton varies. Note that the elliptic fiber is only a book-keeping device, an auxiliary object, which is a priori not part of physical ten-dimensional spacetime. However, we will see in section 4.2 that this understanding can be modified in the context of M-theory.

As a concluding remark note that the above is a sometimes misleading way to think about F-theory. To determine the details of the effective action one has to approach F-theory via its duality to M-theory. The reader might wonder whether a field theoretical description of F-theory, a definition from first principles, exists. To date such a fundamental description is not known.

4.1. F-theory via Type IIB String Theory

Our first approach to F-theory is via type IIB string theory since it is probably the most intuitive one. Type IIB string theory is invariant under an $SL(2,\mathbb{Z})$ transformation which enables us to define F-theory. We will consider the D-brane solutions of ten-dimensional type IIB supergravity and observe that the solution for D7-branes is special: It involves a logarithm which does not approach zero infinitely far away from the brane. This is the starting point for F-theory. The following outline is based on [Hog12].

4.1.1. Type IIB String Theory and $SL(2,\mathbb{Z})$ Invariance

The low-energy effective theory of type IIB string theory is type IIB supergravity. This theory is chiral, meaning that both supersymmetry generators have the same ten-dimensional chirality. We have 32 real supercharges (two Majorana-Weyl spinors) with R-symmetry $SO(2)_R \simeq U(1)_R$. There exists a unique linear representation, a multiplet, of the extended $\mathcal{N} = (2,0)$ Poincaré algebra with spins $\leq$ 2. The bosonic part of its decomposition is a graviton $g_{\mu\nu}$, a 2-form B_2, a real scalar ϕ called dilaton, in the RR-sector the form fields C_0, C_2 and C_4 where C_0 is called axion. The p-form potentials have field strengths:

$$H_3 = dB_2, \quad F_{p+1} = dC_p, \qquad \text{for } p = 0,2,4.$$

F_5 is a self-dual field strength. The p-form fields give rise to Dp-branes with $p = -1,1,3,5,7$ to which C_0, C_2, C_4 can couple electrically or magnetically. Additionally, there is the fundamental string F1 to which B_2 couples. The D1-brane is also called *D-string* and the D(-1)-brane is referred to as *D-instaton*.

Due to the presence of a self-dual field strength, there is no standard covariant action which determines the dynamics completely. However it is possible to write down an action which leads to the correct equations of motion. The duality constraint must be imposed directly on the level of the equations of motion.

After introducing $\tau = C_0 + ie^{-\phi}$ and $G_3 = F_3 - \tau H_3$ one can write the type IIB action in the following form [Blu+07]:

$$S_{IIB} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\left(R - \frac{1}{2}\frac{\partial\tau\partial\bar{\tau}}{\mathrm{Im}(\tau)^2} - \frac{1}{2}\frac{G_3\wedge *G_3}{\mathrm{Im}\tau} - \frac{1}{4}\tilde{F}_5\wedge *\tilde{F}_5\right) + \frac{1}{8i\kappa^2}\int \frac{C_4\wedge G_3\wedge \tilde{G}_3}{\mathrm{Im}\tau} \tag{4.1}$$

This action is manifestly invariant under the $SL(2,\mathbb{Z})$ transformation:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} C_2 \\ B_2 \end{pmatrix}, \quad C_4 \to C_4, \quad \text{with} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),$$

because the field strength $\tilde{F}_5$ is invariant and the other fields transform as:

$$G_3 \to \frac{G_3}{c\tau + d}, \quad \mathrm{Im}(\tau) \to \frac{\mathrm{Im}\tau}{|c\tau + d|^2}, \quad \partial\tau \to \frac{\partial\tau}{(c\tau + d)^2}.$$

Thus we have demonstrated that this symmetry holds in the low-energy effective theory of type IIB string theory. There are strong arguments that this symmetry is actually a symmetry of the whole string theory (see [BLT12]).

4.1.2. Supergravity Brane Solutions and Backreaction

Now we would like to concentrate on D-branes and their solutions in supergravity. As their defining feature D-branes have RR-charge and non-zero tension. Therefore they backreact gravitationally on the geometry and via the standard coupling on the p-form fields. When dealing with these backreactions the standard procedure is to take the probe approximation. This means that one neglects the backreaction of the brane on the geometry far away from the brane.
This reasoning can be motivated by the following argument [Wei10]. Intuitively a D-brane is source for the background fields which we denote symbolically by Φ. Such a source leads to a Poisson equation in the normal directions. For a Dp-brane we have $n = 9 - p$ normal spatial directions. In those normal dimensions the D-brane looks like a point. Therefore the Poisson equation generally reads:

$$\Delta\Phi \sim \delta(r),$$

where r is the distance to the brane. The equation is solved by:

$$\Phi(r) = \begin{cases} \log r & \text{for } n = 2, \\ \frac{1}{r^{n-2}} & \text{for } n > 2. \end{cases} \tag{4.2}$$

We see that for $n > 2$ the probe approximation is justified but for $n = 2$ the argument is not valid! The latter case corresponds to D7-branes. There are two problems. First, $\log r$ does not approach zero for $r \to \infty$ and second, the logarithm introduces a branch cut, i.e. the background field τ is multivalued. Let us examine this phenomenon more explicitly. A D7-brane is an electric source for an 8-form potential. So we have [HW97]:

$$d * F_9 = \delta(z - z_0),$$

where $z = x_8 + ix_9$ is a complex coordinate in the normal directions and the brane is located at $z = z_0$. Integrate the relation:

$$1 = \int_{\mathbb{C}} d * F_9 \overset{\text{Stokes}}{=} \oint_{S^1} *F_9 = \oint_{S^1} F_1 = \int_{S^1} dC_0.$$

We can read off: Encircling the brane once leads to an axion shift $C_0 \to C_0 + 1$ or equivalently $\tau \to \tau + 1$. This is a called a *monodromy*. Note that this monodromy is physically consistent since

this transformation is the usual gauge transformation of the axion.
More generally, we have to consider so-called $[p, q]$*-branes*. These are objects on which (p, q)-strings can end. (p, q)-strings are a generalisation of the fundamental string. The fundamental string is electrically charged under the NSNS-field B_2. Additionally, we have D1-branes in the theory. These are strings which are charged under the RR-field C_2. A (p, q)-string carries p units of electric B_2-charges and q units of electric C_2-charges. These more general strings will induce a full $SL(2, \mathbb{Z})$ monodromy. Note that it is in principle possible to transform a general $[p, q]$-brane into a D7-brane, i.e. $p = 1$ and $q = 0$. However this is only possible locally and not simultaneously for several 7-branes. We will come back to this in section 4.1.3.
One might think that this $SL(2, \mathbb{Z})$ monodromy complicates the physical interpretation of the background. But there is a way out: Recall that Type IIB string theory has an $SL(2, \mathbb{Z})$ symmetry as we have seen at least in the low-energy limit in section 4.1. Hence our multivalued background fields are physically single-valued exploiting the $SL(2, \mathbb{Z})$ symmetry.

4.1.3. Elliptic Fibrations

In the last section we have seen how the monodromy of the logarithm, which comes into play via the two-dimensional Poisson equation, perfectly fits to the symmetry of Type IIB string theory. Thus the multivaluedness of τ is not physical but an artefact of the formulation. Our next aim is to find a formulation for which the $SL(2, \mathbb{Z})$ invariance is *intrinsic*. We follow the lines of [Cec10].
Vafa's crucial idea [Vaf96] was to interpret τ as the complex structure modulus of an elliptic curve. It stands to reason since elliptic curves enjoy a $SL(2, \mathbb{Z})$ symmetry as well. In other words we take our ten-dimensional spacetime X_{10} and attach an elliptic curve to every point of X_{10} such that the complex structure varies over X_{10} in the same way as τ, the axio-dilaton. In the maths literature this construction is called *elliptic fibration*. It is a general fact that elliptic curves can be written in Weierstrass form:

$$y^2 = x^3 + fxz^4 + gz^6, \tag{4.3}$$

where x, y, z are coordinates of $\mathbb{P}^{1,1,2}$, the ambient space of the elliptic curve, and f and g are sections of a line bundle $\mathcal{L}$ of the base:

$$f \in \mathcal{O}(\mathcal{L}^4), \quad g \in \mathcal{O}(\mathcal{L}^6).$$

A standard result in the theory of elliptic curves is that an elliptic curve degenerates whenever

$$\Delta \equiv 4f^3 + 27g^2 \in \mathcal{O}(\mathcal{L}^{12}) \tag{4.4}$$

vanishes. In section 4.3 we will see that the locus where $\Delta = 0$ and the fiber degenerates plays a crucial role in the analysis of F-theory models and their massless spectrum.
In this fashion, we have found a formulation with manifest $SL(2, \mathbb{Z})$ invariance without loosing any information. The advantage of the formulation in terms of elliptic fibrations is that it is very geometric and we can employ the whole machinery of algebraic geometry and topology. In mathematical terms

we have a twelve-dimensional manifold Y_{12} with a natural projection to physical spacetime $\pi : Y_{12} \to X_{10}$. Besides the fibration comes with a preferred section $\sigma : X_{10} \to Y_{12}$ which maps a point of spacetime to the neutral element of the elliptic curve in the fiber viewed as an abelian group. We can identify physical spacetime with the image of σ.

All in all F-theory is a 12-dimensional theory where two dimensions are very different to the other ten in the sense that the elliptic curve exhibits a complex structure in contrast to the spacetime dimensions.

4.2. F-theory via M-theory

So far we have looked at F-theory from the type IIB point of view. The essential idea was to introduce the elliptic fibration as a bookkeeping device to reinterpret the complex scalar field τ called axio-dilaton with its $SL(2,\mathbb{Z})$ invariance as complex structure parameter of elliptic curves. In the next section we would like to widen our understanding by adding a physical interpretation of the elliptic fiber. The M-theory star indicates that type IIB theory is connected to M-theory via type IIA theory. The following section is based on [Den08].

4.2.1. Constant Axio-Dilaton

Let us consider eleven-dimensional supergravity on a torus $T^2 = S^1_A \times S^1_B$ and let the non-compact spacetime directions be flat: $\mathbb{R}^{1,8}$. The radii of the two S^1s in the torus are denoted by R_A and R_B. Taking S^1_A to be the M-theory circle the theory reduces to type IIA on $S^1_B \times R^{1,8}$ in the low-energy limit. The next step is to apply T-duality which gives us type IIB theory on $S^1_{\tilde{B}}$ with $R_{\tilde{B}} = \alpha'/R_B$. If one takes the limit $R_B \to 0$, then the compact dimension of the type IIB theory will uncompactify: $R_{\tilde{B}} \to \infty$. In this way eleven-dimensional supergravity on a vanishing torus is dual to type IIB theory on $\mathbb{R}^{1,9}$.

In the eleven-dimensional theory the limit $R_B \to 0$ can be realized by keeping the complex structure modulus of the torus which is essentially given by $\tau \sim R_A/R_B$ fixed and taking the area of the torus $\text{area}(T^2) \sim R_A \cdot R_B$ to zero.

By explicitly tracing the complex structure modulus of the torus through the KK reduction and the T-duality application one can show that it turns out to be the axio-dilaton in the type IIB picture. Additionally the volume of the torus can be interpreted as the KK mass scale in the type IIB compactification on $\mathbb{R}^{1,8} \times S^1_{\tilde{B}}$. By shrinking the size of the torus the KK modes in type IIB become lighter until they are massless in the zero-volume limit and rearrange in ten-dimensional massless multiplets. Details on this can be found in [Den08].

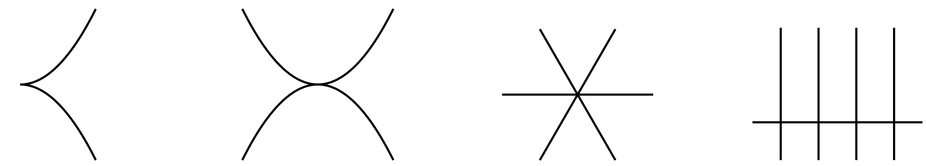
M-theory on $T^2|_{\text{area}(T^2)\to 0,\, \tau \text{ fixed}} \longleftrightarrow$ Type IIB theory with constant axio-dilaton τ.

4.2.2. Varying Axio-Dilaton

One of the essential ingredients of type IIB string theory are D-branes. As we know from the supergravity solution of D-branes (see (4.2)) the axio-dilaton won't be constant as soon as D-branes are present. Thus the M-theory approach to F-theory has to be generalized to non-constant axio-dilaton and therefore non-constant complex structure modulus of the fiber.
For definiteness let us focus on compactifications to six dimensions from now on. The generalization to other even dimensions is immediate.
Consider M-theory on $\mathbb{R}^{1,5} \times Y_3$ where Y_3 is a Calabi-Yau threefold which is elliptically fibered over the base B_2 which shall be a complex compact manifold of complex dimension two. In our cases it will always be $B_2 = \mathbb{CP}^2$. By the same arguments as before we take the area$(T^2) \to 0$ limit but now fiber-wise. Then, one of the two real fibral dimensions will grow and recombine to one new spacetime dimension. The resulting theory is dual to type IIB on $\mathbb{R}^{1,5} \times B_2$. Since we started with a non-trivial elliptic fibration the complex structure modulus τ and therewith the axio-dilaton is varying on B_2 which is what we wanted.
Having this construction in mind we can "define" F-theory. The quotation marks shall emphasise that no fundamental dynamical definition is known to date. We can explore F-theory only through its dualities to other theories. In the type IIB formulation the elliptic fiber is only a bookkeeping device whereas in the M-theory formulation of F-theory one dimension of the elliptic fiber becomes part of the eleven-dimensional physical spacetime.
Through this formulation we have found a powerful tool to analyse F-theory compactifications: The low-energy effective limit of M-theory on Y_3 *without* taking the F-theory limit area$(T^2) \to 0$ is supergravity in $\mathbb{R}^{1,4}$. This theory is connected to F-theory on $\mathbb{R}^{1,5}$ by compactification on a S^1. This correspondence will be heavily exploited in section 4.4.

4.3. Fiber Degenerations

Mathematically an elliptic fibration is defined as a fibration with almost all fibers being smooth curves of genus one. All fibers which are not elliptic curves are called *singular fibers*. They are unions of rational curves which may have singularities or multiple multiplicities. The singular fibers of an elliptic surface were classified by Kodaira [Kod63] [Kod68]. The classification is tightly connected to the classification of semi-simple Lie algebras over algebraically closed fields. The Dynkin diagrams appearing there can be found in the geometry of singular fibers in the intersection pattern of the rational curves after resolution. Each curve corresponds to a node in a Dynkin diagram and every intersection corresponds to an edge. In this way a singular fiber corresponds to a Lie algebra. This is how non-abelian gauge theory is realised in F-theory. In the following the basic ideas of this identification is outlined.
As soon as one allows for a varying axio-dilaton, the elliptic fibration will develop singularities. These singularities come in two different forms. First there exist singularities of the elliptic fibration which are characterised by a singular fiber but leave the total space smooth. An example for this are type I_1 singularities which are characterised by vanishing orders $(f, g, \Delta) = (0, 0, 1)$. They correspond to the presence of a single 7-brane which can be seen as follows. Consider the $SL(2, \mathbb{Z})$-invariant



(a) Type II (b) Type III (c) Type IV (d) Type I_0^*

Figure 4.1.: Some fiber singularity types.

j-function. For our purposes we do not need the proper definition. If suffices to state that it can be expanded like:

$$j(\tau) = \exp(-\pi i \tau) + 744 + \exp(2\pi i \tau) + \dots .$$

For elliptic curves it can be shown that:

$$j(\tau) = \frac{13824\, f^3}{\Delta}.$$

Let $z_1 = 0$ locally describe the position of the brane. Then, j is proportional to $1/z_1$ which follows from the above equations. The above equations imply that τ depends logarithmically on z_1 which is the characteristic sign of a 7-brane (see discussion around (4.2)).

The other class of singular elliptic fibers consists of singularities where not only the fiber but also the total space becomes singular. This happens if Δ vanishes to higher order. For $Y_2 = K2$ Kodaira has classified all possible singular fibers. Here one finds gauge algebras from the ADE-family. However if one considers elliptic fibrations of higher dimensions, one has to take monodromies into account, which act on the singular locus and therefore on the Dynkin diagram. Through this mechanism it is possible to construct singularities with associated Lie algebras of the B- and C-family and the exceptional ones. The monodromies were systematically studied by Tate [Tat75] and are nicely presented in table 4 of [GM12] (see table 4.1).

The fiber singularities which are most relevant in the following are type II, type III, type IV and type I_0^*. They are displayed in figure 4.1.

Table 4.1 has to be read in the following way. f and g are the sections appearing in the Weierstrass form and Δ is the discriminant of the elliptic curves. These quantities are defined around (4.3) and (4.4). In the table $\mathrm{ord}_{\Sigma_1}(f)$ refers to the vanishing order of f along Σ_1 where Σ_1 is the locus in the base over which the singularity is fibered. Likewise $\mathrm{ord}_{\Sigma_1}(g)$ and $\mathrm{ord}_{\Sigma_1}(\Delta)$ are defined. The quantity z_1 is the coordinate which vanishes along Σ_1 (see [GM00] for details).

The monodromy, whenever it is relevant, is described by a polynomial of degree two or three in an auxiliary variable ψ which is a meromorphic section of a suitable line bundle over Σ_1. Most polynomials in the table are of order two. In this case one has to consider the discriminant of the quadratic equation.[1] If the discriminant is a square, i.e. of the form $(\dots)^2$, the monodromy cover is reducible and one gets the larger gauge group. If the square root does not exist, the cover is irreducible and one

[1] The discriminant of the quadratic equation $x^2 + ax + b = 0$ is defined as $a^2 - 4c$. It is the expression which appears under the square root in the formula for the solutions of the equation.

	$(\mathrm{ord}_{\Sigma_1}(f), \mathrm{ord}_{\Sigma_1}(g))$	$\mathrm{ord}_{\Sigma_1}(\Delta)$	Equ. of monodromy cover	Gauge group
I_0	$(\geq 0, \geq 0)$	0	–	–
I_1	$(0,0)$	1	–	–
I_2	$(0,0)$	2	–	$\mathfrak{su}(2)$
$\mathrm{I}_m,\, m \geq 3$	$(0,0)$	m	$\psi^2 + (9g/2f)_{z_1=0}$	$\mathfrak{sp}([\frac{m}{2}])$ or $\mathfrak{su}(m)$
II	$(\geq 1, 1)$	2	–	–
III	$(1, \geq 2)$	3	–	$\mathfrak{su}(2)$
IV	$(\geq 2, 2)$	4	$\psi^2 - (g/z_1^2)\|_{z_1=0}$	$\mathfrak{sp}(1)$ or $\mathfrak{su}(3)$
I_0^*	$(\geq 2, \geq 3)$	6	$\psi^3 + (f/z_1^2)\|_{z_1=0} \cdot \psi + (g/z_1^3)\|_{z_1=0}$	$\mathfrak{g}_2$ or $\mathfrak{so}(7)$ or $\mathfrak{so}(8)$
$\mathrm{I}_{2n-5}^*,\, n \geq 3$	$(2,3)$	$2n+1$	$\psi^2 + \frac{1}{4}(\Delta/z_1^{2n+1})(2z_1 f/9g)^3\|_{z_1=0}$	$\mathfrak{so}(4n-3)$ or $\mathfrak{so}(4n-2)$
$\mathrm{I}_{2n-4}^*,\, n \geq 3$	$(2,3)$	$2n+2$	$\psi^2 + (\Delta/z_1^{2n+2})(2z_1 f/9g)^2\|_{z_1=0}$	$\mathfrak{so}(4n-1)$ or $\mathfrak{so}(4n)$
IV^*	$(\geq 3, 4)$	8	$\psi^2 - (g/z_1^4)\|_{z_1} = 0$	$\mathfrak{f}_4$ or $\mathfrak{e}_6$
III^*	$(3, \geq 5)$	9	–	$\mathfrak{e}_7$
II^*	$(\geq 4, 5)$	10	–	$\mathfrak{e}_8$
non-min.	$(\geq 4, \geq 6)$	≥ 12	–	–

Table 4.1.: Kodaira-Tate classification of singular fibers, monodromy covers and gauge algebras [GM12].

gets the smaller gauge group. This generalises easily to order three in the case of I_0^* singularities.
In the notation we do not distinguish between a Lie algebra and its associated gauge group since the Lie algebra determines uniquely the corresponding connected Lie group.

4.3.1. Enhanced Gauge Symmetry

Since we would like to consider six-dimensional models we need a complex three-dimensional elliptic fibration Y_3 (which has a real four-dimensional base B_2). The next aim is to understand the connection between singular loci in the elliptic fibration and the gauge theory associated to it. To this end we follow the lines of [Wei10].
Let Σ_1 be the divisor, i.e. codimension-one locus, in the base B_2 over which the singularity occurs. In most cases there exists another three-dimensional Calabi-Yau manifold $\tilde{Y}_3$ which is elliptically fibered over B_2 but with the singular fibers over Σ_1 replaced by a tree of $\mathbb{P}^1$s. Let us call them $\mathbb{P}^1_i$ where $i = 1, \ldots, \mathrm{rk}(G)$. The procedure to deform Y_3 into $\tilde{Y}_3$ is called *blow-up along* Σ_1. In doing so, the number of Kähler moduli $h^{1,1}(Y_3)$ increases by the number of inserted $\mathbb{P}^1$s. Additionally one can express the number of Kähler moduli of Y_3 by $h^{1,1}(B_2) + 1$ because the general elliptic fiber has one Kähler modulus. Applying the Shioda-Tate-Wazir theorem,

$$h^{1,1}(\tilde{Y}_3) = \mathrm{rk}(G) + h^{1,1}(B_2) + 1. \tag{4.5}$$

By construction the $\mathbb{P}^1$s are fibered over the divisor in the base Σ_1. So we can define the respective divisor in the total space:

$$D_i : \mathbb{P}^1_i \to \Sigma_1.$$

In other words the D_i are $\mathbb{P}^1$-fibrations over Σ_1. Additionally one can define:

$$D_0 = \Sigma_1 - \sum_i a_i D_i,$$

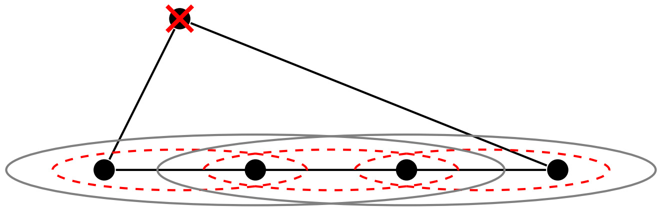


Figure 4.2.: The possibilities how the M2-brane can wrap chains of $\mathbb{P}^1$s in the affine I_4 Dynkin diagram. The crossed out node represents the $\mathbb{P}^1$ which is intersected by the zero section and therefore cannot be wrapped. Additionally the M2-brane can wrap the $\mathbb{P}^1$s individually and all of them at once as well.

where a_i are the Dynkin labels of the associated Lie algebra G for the respective $\mathbb{P}^1$ which will become clear later. D_0 and the D_i define the affine Dynkin diagram of G in the sense advertised above:

$$\int_{\tilde{Y}_3} [D_i] \wedge [D_j] \wedge \omega = -c_{ij} \int_{\Sigma_1} \omega \qquad \forall\, \omega \in H^2(B_2),\ i,j = 1,\ldots,\mathrm{rk}(G),$$

where c_{ij} is the Cartan matrix of G and $[D_i] \in H^2(\tilde{Y}_3)$ are the Poincaré-dual 2-forms of the divisors D_i in $\tilde{Y}_3$.

With the help of F/M-duality one can show that actually a gauge theory with gauge bosons live on the 7-brane. In this duality the M-theory three-form C_3 reduced along the general (non-singular) fiber gives rise to the type IIB two-forms B_2 and C_2. By resolving the singular fibers we introduce new 2-cycles, namely the $\mathbb{P}^1_i$s, along which C_3 can be reduced as well. Integrating the three-form over a $\mathbb{P}^1_i$ gives a one-form gauge potential $A_i = \int_{\mathbb{P}^1_i} C_3$. In total $\mathrm{rk}(G)$ gauge potentials are obtained in this fashion. These are the Z-bosons which are a part of the adjoint representation of the gauge algebra. The rest, i.e. the $W^{\pm}$-bosons, come from the M2-brane of M-theory. It can wrap both single $\mathbb{P}^1_i$s and chains of them: $\mathbb{P}^1_i \cup \mathbb{P}^1_{i+1} \cup \ldots \cup \mathbb{P}^1_j$ with $i \leq j$. This can be done in two orientations. Thus the number has to be multiplied by two. Obviously M2-wrappings on non-vanishing two-cycles give massive particles. However, these particles become massless taking the F-theory limit during which the resolution cycles shrink to zero size. In this way we obtain the $\mathrm{W}^{\pm}$-bosons which complete the adjoint representation of the gauge algebra.

Let us illustrate the above by an example. Consider an elliptic fibration with a codimension-one locus over which the fiber has an A_3 singularity which gives rise to an $SU(4)$ gauge theory. The adjoint representation of $SU(4)$ has 15 dimensions, three of which come from reducing C_3 along the three available $\mathbb{P}^1$s. Additionally the M2-brane can wrap each $\mathbb{P}^1$ individually (3 states), it can wrap chains of two $\mathbb{P}^1$s (2 states) and chains of three $\mathbb{P}^1$s (1 state). This is schematically shown in figure 4.2. These six states correspond to the W^+-bosons. Wrapping with opposite orientation gives the six W^--bosons. In total, we find $3 + 6 + 6 = 15$ states which form the adjoint representation of $SU(4)$.

As a concluding remark note that the above mechanism produces non-abelian gauge groups. Abelian factors are not covered here; they are realised differently. To find the number of abelian factors one has to take global aspects of the fibration into account. Essentially the number of $U(1)$s is given by the dimension of the so-called Mordell-Weil group. Details on this can be found in [MV96a] and [BCV14]. In any way the total rank of the gauge group is given by $h^{1,1}(Y_n) - h^{1,1}(B_{n-1}) - 1$.

4.3.2. Charged Matter

After having dealt with the gauge theory the naturally arising question is: Which mechanism in F-theory is responsible for matter fields that are charged under the gauge group? From the duality to type IIB string theory we expect to find this matter at the transversal intersection of two branes.

When two 7-branes intersect in F-theory the fibration develops a more severe singularity. This is obvious since the singularity type depends crucially on the vanishing order of Δ as can be seen in table 4.1. When two 7-branes intersect the vanishing order of Δ inevitably increases. Formally we can associate a Lie algebra to this codimension-two locus which is bigger than the Lie algebra corresponding to the codimension-one locus. Note that this enhanced Lie algebra does not generate a gauge theory for us because the $\mathbb{P}^1$s are already partly wrapped in order to fill the adjoint representation of the Lie algebra living on the brane. Nonetheless there are new two-cycles at the codimension-loci which can be wrapped by M2-branes. These will organise in representations of the gauge groups living on the two intersecting branes.

Let G_1, G_2 be the two gauge groups associated to the singularities above the two divisors which intersect and let G_{12} be the enhanced gauge group at the intersection locus. In the simplest case one can determine the matter representation by pure group theory: The adjoint representation of G_{12} decomposes into the sum of the adjoint representation of G_1, the adjoint of G_2 and additional representations. This last contribution is the matter charged under both gauge groups.

$$\begin{aligned} &G_{12} \to G_1 \times G_2 \\ &\text{adj.} \to (\text{adj.}, 1) \oplus (1, \text{adj.}) \oplus \text{Matter reps.} \end{aligned}$$

As an example consider an I_5 and an I_1 singularity along two divisors in the base. The two divisors shall intersect at one point. The I_1 locus does not carry a gauge group whereas an $SU(5)$ gauge theory lives on the other brane. At the intersection locus the singularity enhances to I_6, i.e. formally $SU(6)$ (see table 4.1). A quick look into [Sla81] confirms:

$$\begin{aligned} SU(6) &\to SU(5) \\ \mathbf{35} &\to \mathbf{24} \oplus \mathbf{5} \oplus \bar{\mathbf{5}} \oplus \mathbf{1}. \end{aligned}$$

Thus we observe that a fundamental representation lives at this type of enhancement.

For some enhancements it is clear which matter representations live at codimension-two. However, for some cases it is not obvious. As a first example consider a II $\to$ III or a II $\to$ IV enhancement. Since there is no gauge group associated to type II we cannot apply the above ideas. The main objective of this thesis is to find out what happens here. We will go into detail in section 6. In the literature we find examples in [GM00] and [GM12].

Please note that Kodaira's classification of singular fibers does only hold in codimension-one. Thus the fibers may look different at codimension-two: It can happen that the nodes of the Dynkin diagram are partially deleted. We will observe this phenomenon explicitly e.g. in the discussion around figure A.1.

4.4. Mapping Six-dimensional Supergravity to F-theory

Finally we would like to connect the two theories that were presented so far: F-theory on an elliptically fibered Calabi-Yau threefold Y_3 with base B_2 and six-dimensional $\mathcal{N} = (1,0)$ supergravity. The massless spectrum of a F-theory model must fit into multiplets of its low-energy effective theory, in this case six-dimensional supergravity. In other words one has to match geometric properties of the F-theory compactification space with the variables of six-dimensional supergravity, most importantly the number of hyper, vector and tensormultiplets.

The idea behind the identification is to consider M-theory on Y_3 which yields effectively five-dimensional supergravity. Additionally one studies the compactification of six-dimensional supergravity to five dimensions and matches the properties of these to theories explicitly. Taking the F-theory limit means going from M-theory on Y_3 to F-theory on Y_3 and in the low-energy sector going from five to six dimensions. In this way we can exploit our knowledge about M-theory and compactifications to extract the information about F-theory. As always the details are very involved. For our purposes it suffices to state the results of the analysis which is explicitly carried out in [BG12].

The total number of tensor fields is given by:

$$n_T = h^{1,1}(B_2) - 1.$$

In this work we focus on $B_2 = \mathbb{P}^2$. For $\mathbb{P}^2$ the number of Kähler moduli equals one implying no tensormultiplets.

The above identification is very plausible from the type IIB point of view: The RR 4-form C_4 can be reduced along two-cycles (which are counted by $h^{1,1}(B_2)$) leaving effectively two-tensors.

The hypermultiplets come from different sources. First, there are charged hypermultiplets located at codimension-two as we have already described. Second, the complex structure moduli contribute to the hypermultiplets as well:

$$n_{H_0} = \text{CxDef}(Y_3) + 1.$$

One can compute the number of complex structure moduli CxDef(Y_3) with the help of the Hodge number $h^{2,1}(\tilde{Y}_3)$ as long as the singularities are resolvable. If they are more severe one has to add corrections to the naïve $h^{2,1}$ (see the discussion around (5.2) below). The details are described in [AGW16]. Generally, there are also $g(\Sigma_i)$ hypermultiplets for each codimension-one locus with enhanced gauge symmetry with g being the genus of the divisor. However in our cases the genus of the divisors will always be zero.

Finally, we need to identify the vectors a and b_i which come into play in (3.1) and play a crucial role in the anomaly conditions (3.3) to (3.8). One can show (see again [BG12]) that one has to identify:

$$a \longleftrightarrow K_B, \tag{4.6}$$

$$b_i \longleftrightarrow \text{divisor class in the base of the respective gauge brane} \tag{4.7}$$

The scalar product on the vector space is identified with the intersection product in B_2 which is

well-defined since in two dimensions divisors and curves have the same dimension.
Generally speaking the canonical class of $\mathbb{P}^n$ is given by $-(n+1)H$ where H is the hyperplane class. Thus, $a \cdot a$ which appears in (3.5) can be calculated in our F-theory on $\mathbb{P}^2$ context by intersecting $-3H$ with itself. Since $H \cdot H = 1$ we see that $a \cdot a = 9$. This is consistent with our previous result that there are no tensormultiplets in the theory.

5. Some Geometry

In the last section we outlined that codimension-two loci give rise to matter representations. However it is not known whether uncharged matter occurs if there is a type II singularity at codimension one. In the sequel a particular type of F-theory compactifications to six dimensions is analysed in order to learn about these matter representations via the previously developed machinery.

The big-picture idea is the following: By computing the number of complex structure moduli of an elliptically fibered threefold Y_3 with isolated terminal singularities we know for sure how many neutral hypermultiplets are in the theory. Since F-theory models are anomaly-free the spectrum has to satisfy the anomaly constraints in six dimensions. With this tool at hand it is possible to check the correctness of the number of complex structure moduli. Besides the total number of complex structure moduli is divided into two parts: those which leave the terminal singularities untouched and those which deform them. The first part will correspond to unlocalised hypermultiplets and the second one to localised hypers which is the essential statement of this thesis.

Our first task will be to compute the topological Euler characteristic of an elliptic fibration over a two-dimensional base because it can be used to compute the number of complex deformations which controls the total number of uncharged hypermultiplets. This is done in full generality without restricting to the case $B_2 = \mathbb{CP}^2$.

5.1. Complex Structure Deformations and Matter at Codimension Two

The main objective of this thesis is to determine whether there exists uncharged matter at codimension two. Generally the origin of uncharged matter are the complex structure deformations of the total space Y_3. In this section let us omit the index of Y_3 and keep in mind that part of the results hold only for Calabi-Yau *three*folds. If the Calabi-Yau manifold Y is crepant resolvable it is possible to compute $\mathrm{CxDef}(Y)$ via the Hodge number $h^{2,1}(\tilde{Y})$. Concretely, this can be done by applying the identity:

$$\mathrm{CxDef}(Y) = \mathrm{CxDef}(\tilde{Y}) = h^{2,1}(\tilde{Y}) = h^{1,1}(\tilde{Y}) - \tfrac{1}{2}\chi_{\text{top}}(\tilde{Y}). \tag{5.1}$$

However one has to be aware of the fact that the above formula is only valid for crepant resolvable Y. As soon as Y possesses terminal singularities Hodge decomposition breaks down and the formula is not valid any longer.

So let us compute the number of complex structure deformations of such a non-resolvable space and discriminate them into the ones which leave the singularity untouched and the ones which deform the singularity. It is a mathematical fact [AGW16] that if Y is a $\mathbb{Q}$-factorial Calabi-Yau threefold with isolated terminal singularities then it admits a smoothing Y_t and the number of complex structure moduli is given by:

$$\mathrm{CxDef}(Y) = \tfrac{1}{2}\, b_3(Y) + \tfrac{1}{2} \sum_P m_P,$$

where $b_3(Y)$ is the third Betti number, P are the isolated singular points and m_P is the Milnor number of the singularity (see section 5.5). Besides one can show that the number complex structure deformations of a singular Calabi-Yau and a smoothing are the same: $\mathrm{CxDef}(Y) = \mathrm{CxDef}(Y_t)$. It follows that $b_3(Y_t) = b_3(Y) + \sum_P m_P$ since the third Betti number equals CxDef in the smooth case. Thus,

$$\tfrac{1}{2}\, b_3(Y_t) = \tfrac{1}{2}\, b_3(Y) + \tfrac{1}{2} \sum_P m_P = \tfrac{1}{2} \Big(b_3(Y) - \sum_P m_P\Big) + \sum_P m_P.$$

In other words,

$$\tfrac{1}{2}\, b_3(Y_t) - \sum_P m_P = \tfrac{1}{2}\Big(b_3(Y) - \sum_P m_P\Big) = \mathrm{CxDef}(Y) - \sum_P m_P.$$

It turns out that $\sum_P m_P$ on the left-hand side actually counts the number of so-called *versal deformations*, i.e. deformations which destroy the form of the singularity.[1] Put differently the total number of complex structure moduli can be split into two parts: the moduli which preserve the singularity and those which deform the singularity. These considerations lead to the following conjecture:

The total number of uncharged hypermultiplets (both localised and non-localised) is given by $\mathrm{CxDef}(Y)+1$. *A part of these uncharged hypermultiplets is localised (*$\sum_P m_P$*) and the rest is non-localised (*$1 + \mathrm{CxDef}(Y) - \sum_P m_P$*).*

Later we will analyse some models in order to observe at which types of singularities uncharged localised matter lives. To this end we need to be able to explicitly compute $\mathrm{CxDef}(Y)$. The crucial formula in this context is [AGW16]:

$$\mathrm{CxDef}(Y_3) = \mathrm{KaDef}(Y_3) - \tfrac{1}{2}\, \chi_{\text{top}}(\tilde{Y}_3) + \tfrac{1}{2} \sum_P m_P. \tag{5.2}$$

The number of Kähler deformations KaDef is still given by $h^{1,1}(\tilde{Y}_3)$ and m_P is again the Milnor number associated to the point P. The sum goes over all singular codimension-two loci. If a point is smooth the Milnor number vanishes and (5.2) reduces to (5.1). This distribution is visualised in figure 5.1.

[1] This assertion is based on the fact that in our case the Milnor number and the Tyurina number coincide. Mathematical details can be found in [RT08].

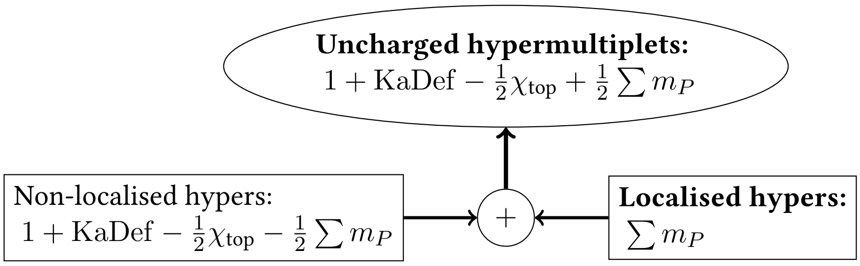


Figure 5.1.: Origin of uncharged hypermultiplets in six-dimensional F-theory compactifications.

5.2. How to compute χ_{top}

To compute the Euler characteristic of an elliptically fibered Calabi-Yau manifold we follow the approach presented in [GM00]. The elliptic fibration is given in Weierstrass form:

$$P_W = -y^2 + x^3 + fxz^4 + gz^6 = 0,$$

where f, g are sections of $\mathcal{O}(-4K_B), \mathcal{O}(-6K_B)$, respectively (such that the total space is Calabi-Yau). We assume that the gauge group does not contain any abelian factors. Besides we assume for simplicity that it has only one semi-simple factor. Later we will generalise the computation for χ_{top} to theories with two identical simple factors (see section 7).

5.2.1. Notation

First we have to introduce a bunch of notation. For a better overview we include figure 5.2 on which most of the notation is outlined.

- The discriminant shall be of the form $\Delta \equiv 4f^3 + 27g^2 \stackrel{!}{=} z_1^m \cdot \sigma_0$ where z_1 is the local coordinate on the base B_2 which defines the divisor Σ_1 on which the 7-brane is wrapped. σ_0 does not vanish identically along $z_1 = 0$, i.e. m is chosen to be maximal. σ_0 locally defines the residual discriminant Σ_0. Over the generic point of Σ_0 a type I_1 singularity is fibered. Thus, there is no gauge group located here.
- Likewise the sections f and g shall vanish to degree μ_f and μ_g along Σ_1. Locally:

$$f = z_1^{\mu_f} f_0 \quad \text{and} \quad g = z_1^{\mu_g} g_0, \tag{5.3}$$

  such that f_0 and g_0 do not vanish along z_1.
- Additionally let $\mu_P(f, g)$ be the intersection multiplicity of f_0 and g_0 at a point P.
- The generic fibers over Σ_0, Σ_1 are called $X_{\Sigma_0}, X_{\Sigma_1}$, respectively.
- Generically Σ_0 and Σ_1 intersect. There is a finite number of groups of intersection points which are locally the same. So let the intersection points be denoted by P_j^i (i labels the groups of locally equivalent points and j enumerates the points in the group). Additionally the residual discriminant will have cuspidal self-intersection points: Q^j. We will suppress the upper index

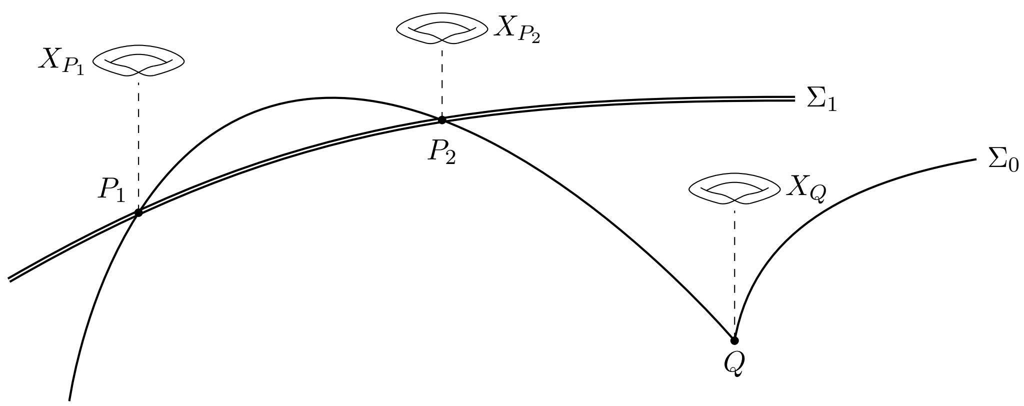


Figure 5.2.: Most of our notation for an elliptically fibered Calabi-Yau threefold with one 7-brane.

of P and Q whenever we want to address the whole group of enhancement points at once. Besides we denote by P the whole set of intersection points in $\Sigma_0 \cap \Sigma_1$.

- The fibers over P_i and Q are called X_{P_i} and X_Q, respectively.
- B_i and C are the number of points P^i and Q, respectively.
- Let $g(\Sigma_1)$ be the genus of the curve[2] Σ_1.
- $(f, g, \Delta)|_\Sigma$ denotes the vanishing orders of f, g, Δ along the general point of a locus Σ. Whenever it is clear which locus we are addressing we will simply write (f, g, Δ).

5.2.2. Contributions to χ_{top}

The topological Euler characteristic has two important properties. First, one can split the space into smaller ones, compute their respective Euler characteristic and then sum all contributions up. This is possible due to the Mayer–Vietoris sequence. Second, for a product space the topological Euler characteristic can be expressed as a product of the Euler characteristics of the factors. In our case the elliptic fibration is locally a product space and therefore we can compute the Euler characteristic of the base and multiply it with the Euler characteristic of the fiber. But since the fibration is generally non-trivial this is only possible locally.
We can split the total space into five components:

1. The fibers over the intersections points P_i: $\bigcup_i \pi^{-1}(P_i)$. Their contribution is

$$\sum_i \chi_{\text{top}}(X_{P_i}) \cdot B_i. \tag{5.4}$$

2. The generic fiber over Σ_1: $\pi^{-1}(\Sigma_1 \backslash P)$. The Euler characteristic of Σ_1 without the enhancement points is given by $\chi_{\text{top}}(\Sigma_1) = 2 - 2g(\Sigma_1)$ minus the number of points P: $\sum_i B_i$. It must be

[2]Note that we are considering Calabi-Yau threefolds with two-dimensional bases. Therefore divisors on the base are curves.

multiplied by the Euler characteristic of the fiber:

$$\chi_{\text{top}}(X_{\Sigma_1}) \cdot \Big(2 - 2\,g(\Sigma_1) - \sum_i B_i\Big). \tag{5.5}$$

3. Analogously there is a contribution from the general fiber over Σ_0:

$$\chi_{\text{top}}\Big(\pi^{-1}(\Sigma_0 \setminus Q \setminus P)\Big). \tag{5.6}$$

4. Finally the fibers over Q contribute:

$$\chi_{\text{top}}(X_Q) \cdot C. \tag{5.7}$$

5. The general fiber over points where the discriminant does not vanish does not contribute to the Euler characteristic since the χ_{top} of a torus is zero.

5.2.3. The Calculation

Let us look at the different contributions in turn. There is nothing to say about (5.4). It is already in its final form. In (5.5) we can replace $\chi_{\text{top}}(X_{\Sigma_1}) = m$. This can be shown by inspecting all possible fiber types, i.e. all Dynkin diagrams. In (5.7) we can set $\chi_{\text{top}}(X_Q) = 2$ because at the points Q the vanishing orders are $(\mu_f, \mu_g, m)|_Q = (1, 1, 2)$ which corresponds to a cuspidal singularity (type II in Kodaira's classification). A cuspidal curve has $\chi_{\text{top}} = 2$ (more on the calculation of χ_{top} for singular elliptic fibers in section 5.4 below). What remains is to compute the number of cusps of Σ_0 and to bring the third contribution into a nicer form.

The number of cusps C. Cusps appear as soon as both f and g vanish along $\Sigma_0 \setminus P$. Since f and g are of the form (5.3) only f_0 and g_0 can vanish along Σ_0 away from Σ_1. The number of intersection points naïvely is $(-4K_B - \mu_f \Sigma_1) \cdot (-6K_B - \mu_g \Sigma_1)$. However we have to correct it by the intersection multiplicity $\mu_{P_i}(f, g)$ of f_0 and g_0 at the points P_i. All in all the number of cusps is given by:

$$C = 24K_B^2 + \Big(4\mu_g + 6\mu_f\Big) K_B \cdot \Sigma_1 + \mu_f \mu_g \Sigma_1^2 - \sum_i \mu_{P_i}(f, g) B_i.$$

The third contribution. First note that along Σ_0 the vanishing order structure is $(f, g, \Delta)|_{\Sigma_0} = (0, 0, 1)$. Therefore the elliptic curve develops a type I_1 singularity. Its Euler characteristic is $\chi_{\text{top}}(X_{\Sigma_0}) = 1$.

If Σ_0 was a smooth curve, its topological Euler characteristic would be given by: $-(K_B + \Sigma_0) \cdot \Sigma_0$. However it can happen that Σ_0 is not smooth at the intersection points P and it is definitely not smooth at the self-intersection points Q. Moreover one has to exclude all these points because they are already taken into account in (5.4) and (5.7). Therefore we have to include correction terms for each point P_1, P_2 and Q:

$$\chi_{\text{top}}\Big(\pi^{-1}(\Sigma_0 \setminus Q \setminus P_1 \setminus P_2)\Big) = \Big(-(K_B + \Sigma_0)\cdot\Sigma_0 + \sum_i \epsilon_i B_i + \epsilon_c C\Big)\cdot \underbrace{\chi_{\text{top}}(X_{\Sigma_0})}_{=1}.$$

Obviously ϵ_i has to be defined such that it is -1 if the considered point is a smooth point on the curve.

There are two tasks left. First, we calculate $-(K_B + \Sigma_0)\cdot\Sigma_0$. Then, we will take a close look at the definition and computation of the ϵ_is.

The Calabi-Yau condition for an elliptic fibration $\Delta \in \mathcal{O}(-12K_B)$ implies: $\Sigma_0 = -12K_B - m\Sigma_1$. It immediately follows that $\Sigma_0 \cdot \Sigma_1 = -12K_B \cdot \Sigma_1 - m\Sigma_1^2$. Applying these two relations several times gives:

$$-(K_B + \Sigma_0)\cdot\Sigma_0 = -132\,K_B^2 - 23\,m\,K_B\cdot\Sigma_1 - m^2\,\Sigma_1^2.$$

Finally, let us properly define the ϵ_is. As having observed ϵ of a smooth point must take the value -1. Let us consider a curve D with singular point $P \in D$. $\phi_1 : B_1 \to B$ shall be the blow-up of the point P with exceptional divisor E. We define the quantity α_1 via $D_1 = \phi_1^*(D) - \alpha_1(P)E$ where D_1 is the strict transform of the curve.[3] Then the Euler characteristic of the blown-up curve is [GM00]:

$$\chi_{\text{top}}(D_1) = -(K_{B_1} + D_1)\cdot D_1 = -(K_B + D)\cdot D - \alpha_1(P)\cdot(\alpha_1(P) - 1).$$

We perform successive blow-ups until the point P is smooth. Since we do not want to include the singular point itself in our calculation of χ_{top} we have to subtract the number of preimages of P under the total blow-up ϕ. We combine the total correction of χ_{top} due to the singular point P into the definition of ϵ:

$$\epsilon := \sum_i \alpha_i(P)\cdot\Big(\alpha_i(P) - 1\Big) - \#\phi^{-1}(P),$$

where i runs over the successive blow-ups one has to perform until the singularity is smoothed out completely.[4] In section 5.3 we will explicitly compute ϵ_i for all singularity types which appear in the context of this thesis.

[3] To illustrate the definition of α_1 let us look at a simple example: Let the curve D be given by the equation $x^3 + y^3 = 0$. It is singular at $(0,0)$. The blow-up $x \to xy, y \to y$ leads to $y^3 \cdot (x^3 + 1) = 0$. Then y is the exceptional divisor of the blow-up and appears with multiplicity $\alpha_1 = 3$.

[4] In our above example $\#\phi^{-1}(P) = 3$ and therefore $\epsilon = 3\cdot 2 - 3 = 3$. Let us look at another example: $x^3 + y^5 = 0$. The first blow-up is $x \to xy$: $y^3 \cdot (x^3 + y^2) = 0$. So $\alpha_1 = 3$. Then perform $y \to xy$: $x^2 \cdot (x + y^2) = 0$. Since $x + y^2$ has only one solution $\#\phi^{-1}(P) = 1$ and $\epsilon = 3\cdot 2 + 2\cdot 1 - 1 = 7$.

5.2.4. The Result

Putting everything together we arrive at our final expression of the topological Euler characteristic of an elliptically fibered Calabi-Yau threefold Y_3 over B_2 with singular locus $\Sigma_0 \cup \Sigma_1$ as specified above:

$$\begin{aligned} \chi_{\text{top}}(\pi : Y_3 \to B_2) = \chi_{\text{top}}\Big(\bigcup_i \pi^{-1}(P_i)\Big) + \chi_{\text{top}}\Big(\pi^{-1}(\Sigma_1 \setminus P)\Big) \\ + \chi_{\text{top}}\Big(\pi^{-1}(\Sigma_0 \setminus Q \setminus P_1 \setminus P_2)\Big) + \chi_{\text{top}}\Big(\pi^{-1}(Q)\Big) \\ = \Big(\sum_i B_i \cdot \chi_{\text{top}}(X_{P_i})\Big) + m\,\Big(2 - 2g - \sum_i B_i\Big) \\ - 132K_B^2 - 23\,m\,K_B \cdot \Sigma_1 - m^2\,\Sigma_1^2 + 3\,C + \sum_i \epsilon_i B_i, \end{aligned} \tag{5.8}$$

with

$$C = 24K_B^2 + \Big(4\mu_g + 6\mu_f\Big)K_B \cdot \Sigma_1 + \mu_f \mu_g \Sigma_1^2 - \sum_i \mu_{P_i}(f, g) B_i. \tag{5.9}$$

5.3. How to compute ϵ

In the last section we defined the quantity ϵ which is an integer associated to a point on a plane complex curve. In the following we will encounter three different classes of points for which we have to calculate ϵ: smooth points, singularities of the form $x^2 + y^n = 0$ for $n \geq 2$ and singularities of the form $x^3 + y^n = 0$ for $n \geq 3$. We drop the index which counts the number of successive blow-ups because it should be clear from the context.

Claim: For smooth points: $\epsilon = -1$.

As we already pointed out there is no need to blow up smooth points ($\alpha = 0$ and $\#\phi^{-1} = 1$). Thus, $\epsilon = \alpha(\alpha - 1) - \#\phi^{-1} = -1$. #

Claim: For singularities of the form $x^2 + y^n = 0$ for $n \geq 2:\ \epsilon = n - 2$.

First consider $x^2 + y^2 = 0$. After the blow-up $x \to xy$ it takes the form $y^2(x^2 + 1) = 0$, i.e. $\alpha = 2$ and $\#\phi^{-1} = 2$. Thus, $\epsilon = 0$ in this case. ✓
Next consider $x^2 + y^3 = 0$. The blow-up $x \to xy$ leads to $y^2(x^2 + y) = 0$ which means $\alpha = 2$ and $\#\phi^{-1} = 1$. Thus, $\epsilon = 1$.✓
Finally perform the induction step. The curve $x^2 + y^{n+2} = 0$ is blown up to $y^2(x^2 + y^n) = 0$ where we set $x \to xy$. Thus, $\alpha = 2$ and $\epsilon_{x^2+y^{n+2}} = 2 + \epsilon_{x^2+y^n}$ which proves the assertion. #

Claim: For singularities of the form $x^3 + y^n = 0$ for $n \geq 3:\ \epsilon = 2n - 3$.

We can show this claim by induction as well. However this time we need three initial steps. First, $x^3 + y^3 = 0$ is blown up to $y^3(x^3 + 1) = 0$ ($x \to xy$), $\alpha = 3$, $\#\phi^{-1} = 3$ and $\epsilon = 3 \cdot 2 - 3 = 3$. ✓
Second, $x^3 + y^4 = 0$ is blown up to $y^3(x^3 + y) = 0$ ($x \to xy$), $\alpha = 3$, $\#\phi^{-1} = 1$ and

$\epsilon = 3 \cdot 2 - 1 = 5$. ✓
Third, $x^3 + y^5 = 0$ is blown up to $y^3(x^3 + y^2) = 0$ ($x \to xy$) and $\alpha = 3$. Since we are free to swap $x \leftrightarrow y$ we already showed that $\epsilon_{x^2+y^3} = 1$. All in all, $\epsilon = 3 \cdot 2 + \epsilon_{x^2+y^3} = 7$.
The induction step is done in steps $n \to n+3$. Consider the curve $x^3 + y^{n+3} = 0$. After the blow-up $x \to xy$, the exceptional divisor y^3 factors: $y^3(x^3+y^n) = 0$. Thus, $\epsilon_{x^3+y^{n+3}} = 3 \cdot 2 + \epsilon_{x^3+y^n} = 6 + 2n - 3 = 2(n+3) - 3$ where we inserted the induction hypothesis $\epsilon_{x^3+y^n} = 2n - 3$. ✓ #

Let us summarize our results: The characteristic number ϵ for a plane curve singularity can be calculated with the help of the following formulae:

$$\epsilon_{\text{smooth}} = -1, \qquad \epsilon_{x^2+y^n,\, n \geq 2} = n - 2, \qquad \epsilon_{x^3+y^n,\, n \geq 3} = 2n - 3. \tag{5.10}$$

5.4. How to compute $\chi_{\text{top}}(X_{P_i})$

The last non-trivial element in formula (5.8) is the topological Euler characteristic of the fiber over the enhancement points P_i. Resolving the fiber leads to several $\mathbb{P}^1$s intersecting each other. The general prescription to compute $\chi_{\text{top}}(X_P)$ is:

1. Take a look at every component, i.e. every $\mathbb{P}^1$ of the fiber. A $\mathbb{P}^1$ has $\chi_{\text{top}} = 2$. Subtract one for every intersection point on the $\mathbb{P}^1$.
2. Add the contributions from all $\mathbb{P}^1$s up.
3. Add one for every intersection point.

Let us make the prescription more concrete by considering some examples. The numbers in parenthesis denote the contributions to χ_{top}. For a visual impression of the fiber types see figure 4.1.

- The type II fiber. It has one component and one singular point (2-1+1). Thus, $\chi_{\text{top}}(\text{type II}) = 2$.
- The type III fiber. It has two components each of which has one singular point $(1 + 1)$. In total there is one singular point (1). Thus, $\chi_{\text{top}}(\text{type III}) = 1 + 1 + 1 = 3$.
- The type IV fiber. It has three components all of which have a singular point $(1 + 1 + 1)$. These three singular points are coincident (1). So, $\chi_{\text{top}}(\text{type IV}) = 3 + 1 = 4$.
- The type I_0^* fiber. It has four components with one singular point $(1+1+1+1)$, one component with four singular points $(2-4)$ and all in all four singular points (4). Therefore, $\chi_{\text{top}}(\text{type I}_0^*) = 6$.

Crucial Remark. It is tempting to determine the fiber type with the help of table 4.1 by plugging the naïve value for the fiber as computed above into formula (5.8). However, this is *not* possible. The reason is that the Tate-Kodaira classification only holds for singular fibers in codimension-*one*. In contrast we look at singularities in codimension-*two*. It turns out that it is possible that parts of the resolved elliptic fiber are deleted. Whenever $\mathbb{P}^1$s are deleted the value of χ_{top} will change. This phenomenon will explicitly appear later in this thesis.

5.5. How to compute m_P

Finally we would like to formally introduce the Milnor number which is associated to a singularity. Let $\mathcal{O}$ be the ring of function germs $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Let $f \in \mathcal{O}$. Then the *Jacobian ideal* J_f *of* f is defined as:

$$J_f = \left\langle \frac{\partial f}{\partial z_i}, 1 \leq i \leq n \right\rangle.$$

J_f is an ideal in $\mathcal{O}$ viewed as an algebra. Therefore the following local algebra is well-defined:

$$\mathcal{A}_f = \mathcal{O}/J_f.$$

It can be shown that $\mathcal{A}_f$ is not only an algebra but also a $\mathbb{C}$-vector space which may or may not be finite-dimensional. One can show that it is finite if and only if the origin is an isolated critical point, i.e. there is a neighbourhood of the origin such that it is the only critical point of f inside that neighbourhood (see below for examples). Still we can define the *Milnor number* m_P *of* f *at the origin* by:

$$m_P = \dim_{\mathbb{C}} \mathcal{A}_f.$$

Examples Let us go through a few examples to develop some intuition for the Milnor number. First consider $\mathbb{C}^2$ with coordinates x, y as embedding space.

- Let $f = xy$. Then the only critical point on $f = 0$ is the origin. Particularly the origin is an isolated critical point and the Milnor number is expected to be finite. The Jacobian ideal is $J_f = \langle x, y \rangle$. Therefore $\mathcal{A}_f = \langle 1 \rangle$ and the Milnor number associated to this singularity is $m_P = 1$.
- Let $f = x^2$. Then the whole y-axis is critical and the local algebra $\mathcal{A}_f$ is expected to be a infinite-dimensional vector space. Since the Jacobian ideal is $J_f = \langle x \rangle$ our algebra is:

  $$\mathcal{A}_f = \mathcal{O}/J_f = \mathbb{C}[[x, y]]/\langle x \rangle = \mathbb{C}[[y]].$$

  The algebra $\mathbb{C}[[y]]$ is infinite-dimensional as a $\mathbb{C}$-vector space and thus the Milnor number is ∞.

In our context we will encounter the following type of singularity. Consider $\mathbb{C}^4$ with coordinates z, x_1, x_2, x_3 as embedding space. Let $f = z^a + x_1^2 + x_2^2 + x_3^2$ with a being an integer ≥ 2. Then,

$$\mathcal{A}_f = \mathcal{O}/\langle z^{a-1}, x_1, x_2, x_3 \rangle = \langle 1, z, z^2, \ldots, z^{a-2} \rangle.$$

Since the number of generators of $\mathcal{A}_f$ is $a - 1$ the Milnor number is $m_P = a - 1$.

6. Models With One 7-brane

Let us turn to the actual models and analyse F-theory compactifications on an elliptically fibered Calabi-Yau threefold Y_3 over the base $B_2 = \mathbb{P}^2$. The considered models are chosen such that we will learn what happens at singularities in codimension two. Generally speaking the complex deformations of the total space Y_3 are responsible for the uncharged hypermultiplets in six-dimensional low energy effective theory. These complex deformations split into two parts: the unlocalised uncharged hypermultiplets and the localized uncharged hypers at codimension two. It will be shown that the number of the localized uncharged hypers is controlled by the so-called *Milnor number* associated to the singularity at codimension two. As stated before the anomaly conditions in six dimensions provide a powerful tool to validate our considerations.

The outline of this chapter is: First we confirm the formulae of the preceding chapter by having a look at smooth or fully resolvable models. Then the famous I_1 conifold model is analysed with our tools and the explicit distribution of uncharged hypermultiplets is stated. Finally we apply our conjecture to models which cannot be fully resolved first with trivial and then with non-trivial gauge group.

To do this we need to specify formula (5.8) to the base $\mathbb{P}^2$: The canonical bundle of $\mathbb{P}^2$ is given by $K_B = -3H$ where H is the hyperplane class. Σ_1 will always be defined by setting a coordinate to zero: $z_1 = 0$. Thus it is in the hyperplane class H. In complex projective space we have the following general relation: $H \cdot H = 1$. Thus,

$$K_B^2 = 9, \quad K_B \cdot \Sigma_1 = -3, \quad \Sigma_1^2 = 1.$$

Additionally, since Σ_0 is in $-12K_B - m\Sigma_1$ we can simplify:

$$\Sigma_0 \cdot \Sigma_1 = (-12K_B - m\Sigma_1) \cdot \Sigma_1 = 36 - m.$$

Moreover, if Σ_1 is defined by $z_1 = 0$ its genus is $g(\Sigma_1) = 0$. With these preliminary considerations we are able to simplify equation (5.8) to:

$$\begin{aligned}\chi_{\text{top}}(\tilde{Y}_3) = -540 + \sum_i B_i \Big(\chi_{\text{top}}(X_{P_i}) + \epsilon_i - 3\mu_i(f,g)\Big) + m \cdot \Big(71 - m - \sum_i B_i\Big) \\ -36\mu_g - 54\mu_f + 3\mu_f\mu_g.\end{aligned} \tag{6.1}$$

6.1. Smooth and Resolvable Models

As a sanity check we consider the smooth and some resolvable models. In this case we can use $\chi_{\text{top}}(\tilde{Y}_3)$ to calculate the number of complex structure moduli,

$$h^{2,1}(Y_3) = h^{1,1}(\tilde{Y}_3) - \tfrac{1}{2}\chi_{\text{top}}(\tilde{Y}_3), \tag{6.2}$$

where $h^{1,1}(\tilde{Y}^3) = \text{rk}(G) + h^{1,1}(\mathbb{CP}^2) + 1 = \text{rk}(G) + 2$ as we know from equation (4.5) where G is the gauge group associated to the singularity over Σ_1 and $\tilde{Y}_3$ is the resolved threefold. Then the number of uncharged hypermultiplets is given by:

$$n_{H_0} = h^{2,1}(Y_3) + 1 = \text{rk}(G) - \tfrac{1}{2}\chi_{\text{top}}(\tilde{Y}_3) + 3.$$

All results of this section are outlined in table 6.1.

6.1.1. The General Weierstrass Model

We start out with the general Weierstrass model, i.e. $\mu_f = 0 = \mu_g$. Then there is no divisor Σ_1 ($m = 0$) and therefore no enhancement points P. All contributions in equation (6.1) vanish except for the first one. Thus, $\chi_{\text{top}}(\tilde{Y}_3) = -540$. The number of uncharged hypermultiplets is 273 which satisfies the anomaly condition since there does not exist any gauge group.

6.1.2. A III $\to$ IV model

Let us set $\mu_f = 1, \mu_g = 2$. Then a type III singularity is fibered over Σ_1. Therefore there is an $SU(2)$ gauge group present in this theory. The discriminant factors in the following fashion:

$$\Delta = z_1^3 \cdot (4f_0^3 + 27g_0^2 z_1).$$

So the codimension two enhancement points are located at $z_1 = 0 = f_0$. Since $f_0 \in \mathcal{O}(11)$ the number of enhancement points is $B_1 = 11$. The behaviour of Σ_0 near P_1 can be locally described by $f_0^3 + z_1 = 0$ which is smooth and $\epsilon_1 = -1$. Since f_0 and g_0 are generic they do not vanish at the enhancement points which means that the intersection multiplicity $\mu_1(f, g)$ vanishes. The topological Euler characteristic of the fiber over points P_1 is given by 4 since the fiber is a full type IV fiber indeed without any nodes deleted. For details see appendix A. All in all, $\chi_{\text{top}}(Y_3) = -456$ and the rank of our gauge group is $\text{rk}(G) = 1$ which gives $n_{H_0} = 232$. The adjoint representation of $SU(2)$ has dimension $n_V = 3$. Thus, $273 - (232 - 3) = 44$ hypermultiplets are needed to cancel the anomaly.
Since a gauge group is present the hypermultiplets have to organise in representations of it. In the case at hand there must be four states per locus which means that there live two copies of the fundamental representation at every codimension-two locus. In appendix A we resolve the model explicitly and show how these two fundamental representations per codimension-two locus can be understood in terms of M2-wrappings. All results are in perfect match with [GM00] in which the same model has been analysed.

6.1.3. A III → I_0^* Model

In this model we set $\mu_f = 1, \mu_g = 3$. The effect compared to the previous model is that the fiber enhances from type III to type I_0^*. The discriminant has essentially the same form,

$$\Delta = z_1^3 \cdot (4f_0^3 + 27g_0^2 z_1^3).$$

such that $B_1 = 11$. In fact all contributions to $\chi_{\text{top}}(\tilde{Y}_3)$ remain the same except for $\chi_{\text{top}}(X_{P_1})$ and ϵ_1. Taking a look at the residual discriminant we see that $\epsilon_1 = 3$ (see (5.10)). In order to compute $\chi_{\text{top}}(X_{P_1})$ one has to resolve the model, which is also done in appendix A. Then the proper transform PT at the singular locus takes the form:

$$PT|_{e_0 \to 0, a_4 \to 0} = y^2 - e_1 f_1 x^3,$$
$$PT|_{e_1 \to 0, a_4 \to 0} = y^2.$$

The fiber has only two components which intersect in one point. It is a type I_0^* fiber with three nodes deleted. This means that the topological Euler characteristic is now 3. In (7.1) all changes compared to the previous model cancel, $\chi_{\text{top}}(\tilde{Y}_3)$ remains the same and by implication $n_{H_{\text{uncharged}}}$ does so as well. Note that the M2-wrapping at codimension-two works exactly the same way as in the previous model since the curve $PT|_{e_1 \to 0, a_4 \to 0}$ is a non-reduced object of multiplicity two and can be wrapped two times.
All in all this model is very similar to the one presented in section 6.1.2 as it has the same number of uncharged and charged hypermultiplets and the same gauge group.

6.1.4. A IV → I_0^* Model

For this enhancement type we have to choose $\mu_f = 2 = \mu_g$. Afterwards this model works completely analogous to the preceding ones. The only issue is the value of $\chi_{\text{top}}(X_{P_1})$. The naïve value for a type I_0^* fiber is $\chi_{\text{top}}(I_0^*) = 6$. It is shown in appendix A that this is actually the correct choice. The singularity characteristic number ϵ_1 has been computed in [GM00] and is given by $\epsilon_1 = 2$. The gauge group of the model is $SU(3)$, i.e. $\text{rk}(G) = 2$ and $n_V = 8$. Putting things together one observes that the anomaly vanishes if there are three fundamental representations located at each codimension-two enhancement point. This is in perfect agreement with the results in [GM00].

6.1.5. Gauge Anomalies

As we know from chapter 3 six-dimensional supergravity does not only suffer from gravitational but also from gauge and mixed anomalies. Let us finally check if the found spectrum respects the anomaly cancellation constraints coming from these anomalies, too. We will show that the found charged part of the spectra (see table 6.1) is already determined by the following assumptions: Consider $\mathbb{P}^2$ as base space for the elliptic fibration. Assume that there are $B_1 = 11, 8$ codimension-two enhancement points, respectively, and that only fundamental representations appear. The gauge group shall be $SU(2), SU(3)$, respectively. With these assumptions we can deduce that there must be two funda-

(μ_f, μ_g)	$(0,0)$	$(1,2)$	$(1,3)$	$(2,2)$
m	—	3	3	4
Enhancements	—	III → IV	III → I_0^*	IV → I_0^*
Gauge Group	—	$SU(2)$	$SU(2)$	$SU(3)$
$n_V = \dim(G)$	—	3	3	8
$\text{rk}(G)$	—	1	1	2
$h^{1,1}(\tilde{Y}_3)$	2	3	3	4
B_1	0	11	11	8
$\chi_{\text{top}}(X_{P_1})$	—	4	3	6
ϵ_1	—	-1	3	2
$\chi_{\text{top}}(\tilde{Y}_3)$	-540	-456	-456	-408
UnlocUnch Hypers n_{H_0}	273	232	232	209
LocCh Hypers $n_{H_{ch}}$	0	44	44	72
Representation	—	2×fund.	2×fund.	3×fund.
$273 - (n_{H_0} + n_{H_{ch}} - n_V)$	0	0	0	0

Table 6.1.: Smooth or resolvable models.

mental representations located at each codimension-two locus in the $SU(2)$, $SU(3)$ case, respectively. To this end the constraints (3.4) to (3.8) are combined with the F-theory-supergravity identifications (4.6) and (4.7). Since our base is $\mathbb{P}^2$ the vector a is identified by $K_{\mathbb{P}^2} = -3H$ where H is the hyperplane class. The index i runs over the non-abelian factors of the gauge group. Here we have $i = 1$ because our models have only one gauge group factor. Thus we can drop the index i in this case for better readability. b is identified with the cohomology class of Σ_1. Since there is only one gauge group factor the constraint (3.8) is not present. Let us plug in the group theoretic quantities A_R, B_R and C_R which are displayed in table 6.2. Moreover $\lambda_{SU(n)} = 1$ and $n_{\text{fund.}} = 2\,B_1$ for the $SU(2)$ case and $n_{\text{fund.}} = 3\,B_1$ for the $SU(3)$ case. In the $SU(2)$ case we have:

$$
\begin{aligned}
9 = (-3H)\cdot(-3H) = a\cdot a &\overset{?}{=} 9 - n_T = 9\ \checkmark,\\
-3 = (-3H)\cdot H = a\cdot b &\overset{?}{=} \frac{\lambda}{6}\left(A_{Adj.} - \sum_R n_R A_R\right) = \tfrac{1}{6}\cdot(4 - 2\cdot 11\cdot 1) = -3\ \checkmark,\\
1 = H\cdot H = b\cdot b &\overset{?}{=} \frac{\lambda^2}{3}\left(\sum_R n_R C_R - C_{Adj.}\right) = \tfrac{1}{3}\cdot(2\cdot 11\cdot \tfrac{1}{2} - 8) = 1\ \checkmark.
\end{aligned}
$$

The constraints in the $SU(3)$ case are:

$$
\begin{aligned}
9 = (-3H)\cdot(-3H) = a\cdot a &\overset{?}{=} 9 - n_T = 9\ \checkmark,\\
-3 = (-3H)\cdot H = a\cdot b &\overset{?}{=} \frac{\lambda}{6}\left(A_{Adj.} - \sum_R n_R A_R\right) = \tfrac{1}{6}\cdot(6 - 3\cdot 8\cdot 1) = -3\ \checkmark,\\
1 = H\cdot H = b\cdot b &\overset{?}{=} \frac{\lambda^2}{3}\left(\sum_R n_R C_R - C_{Adj.}\right) = \tfrac{1}{3}\cdot(3\cdot 8\cdot \tfrac{1}{2} - 9) = 1\ \checkmark.
\end{aligned}
$$

	A_R	B_R	C_R
Fund.	1	0	1/2
Adj.	4	0	8

(a) $SU(2)$

	A_R	B_R	C_R
Fund.	1	0	1/2
Adj.	6	0	9

(b) $SU(3)$

Table 6.2.: The group theoretic constants A_R, B_R and C_R for $SU(2)$ and $SU(3)$.

All in all, we see that gauge anomaly cancellation requires two and three copies of the fundamental representation at each locus in the $SU(2)$ and $SU(3)$ case, respectively. This is exactly what we found in the calculations which have led to table 6.1.

6.2. The I_1 Conifold Model

After this warm-up exercise we are ready to face our first non-resolvable model. We consider the I_1 conifold Tate model since we know from e.g. [BCV14] that there is exactly one uncharged hypermultiplet located at the $I_1 \to I_2$ enhancement points [BCV14].

The appropriate Tate model has vanishing orders $(a_1, a_2, a_3, a_4, a_6)|_{z_1 \to 0} = (0, 0, 1, 1, 1)$ along z_1 which describes the divisor Σ_1 in the base. In this model the discriminant Δ splits into two parts Σ_0 and Σ_1:

$$\Delta = \tfrac{1}{16}\, z_1 \left(a_6\, (a_1^2 + 4\, a_2)^3 + z_1 \cdot (\ldots)\right).$$

Thus there are two types of codimension-two enhancement points: $\{z_1 = 0\} \cap \{a_1^2 + 4\, a_2 = 0\}$ (called P_1) and $\{z_1 = 0\} \cap \{a_6 = 0\}$ (called P_2). There are six points of type P_1 and 17 points of type P_2. At P_1 the fiber enhances from I_1 to II, i.e. $\chi_{\text{top}}(X_{P_1}) = 2$. At P_2 there exists no small resolution (see [GM00]). Therefore, we set $\chi_{\text{top}}(X_{P_2}) = \chi_{\text{top}}(\text{type } I_1) = 1$. The intersection multiplicity $\mu_{P_1} = 2$, $\mu_{P_2} = 0$ and $\epsilon_1 = -1 = \epsilon_2$ (table 4 in [GM00]). All in all, $\chi_{\text{top}}(\tilde{Y}_3) = -523$.

At this point note that this model is not resolvable because there is no gauge group associated to the I_1 fiber. Therefore it remains singular and one has to apply the more general formula to compute the number of complex structure deformations CxDef (see equation (5.2)).

In the case at hand the singularity at P_1 has the local form $z^2 + x_1^2 + x_2^2 + x_3^2 = 0$ which has Milnor number 1 (see section 5.5). Thus the number of complex structure deformations is:

$$\text{CxDef}(Y_3) = 2 + \tfrac{1}{2} \cdot 523 + \tfrac{1}{2} \cdot 17 \cdot 1 = 272.$$

As always there is also the universal hypermultiplet such that $n_H = 273$ which completes the anomaly condition.

We observe that the Milnor number at the codimension-two singularity coincides with the number of expected uncharged localised hypermultiplets. Thus this model is a strong indicator that our assertion in section 5.1 that the number of uncharged localised hypers is given by the Milnor number of the respective singularity is true.

(μ_f, μ_g)	$(1,1)$	$(2,1)$
m	2	2
Enhancements	II $\to$ III	II $\to$ IV
$h^{1,1}(\tilde{Y}_3)$	2	2
B_1	17	17
$\chi_{\text{top}}(X_{P_1})$	2	2
ϵ_1	-1	2
$\chi_{\text{top}}(\tilde{Y}_3)$	-506	-506
a	3	3
m_P	2	2
UnlocUnch Hypers n_{H_0}	239	239
LocUnch Hypers $n_{H_{LocUnch}}$	34	34
$273 - (n_{H_0} + n_{H_{LocUnch}})$	0	0

Table 6.3.: Non-resolvable models with trivial gauge group. a characterises the form of the codimension-two singularity: $z^a + x_1^2 + x_2^2 + x_3^2 = 0$. m_P denotes the corresponding Milnor number.

6.3. Non-resolvable Models with Trivial Gauge Group

In this section we would like to consider two models with trivial gauge group. We chose a type II elliptic curve fibered over the divisor Σ_1. It shall enhance to type III and type IV. This is achieved by setting $\mu_f = 1 = \mu_g$ for the first model and $\mu_f = 2, \mu_g = 1$ for the second model. The discriminants are given by:

$$\Delta_1 = z_1^2 \cdot \left(27g_0^2 + 4z_1 f_0^3\right)$$
$$\Delta_2 = z_1^2 \cdot \left(27g_0^2 + 4z_1^4 f_0^3\right)$$

With our knowledge about ϵ we directly observe $\epsilon = -1$ in the first model and $\epsilon = 2$ in the second model. The intersection multiplicity $\mu(f,g)$ vanishes again since f_0, g_0 are generic. The topological Euler characteristic of the fiber over the enhancement points is $\chi_{\text{top}}(X_{P_1}) = 2$, the χ_{top} of the type II fiber, since both models are not resolvable. With the help of (6.1) we find $\chi_{\text{top}}(\tilde{Y}_3) = -506$ for both models.

The singularity at the codimension-two loci is of the form $z^3 + x_1^2 + x_2^2 + x_3^2 = 0$ which has Milnor number 2. Thus, the number of complex structure deformations is:

$$\text{CxDef}(Y_3) = 2 + \tfrac{1}{2} \cdot 506 + \tfrac{1}{2} \cdot 17 \cdot 2 = 272.$$

As always there is also the universal hypermultiplet such that $n_H = 273$ which completes the anomaly condition. These 273 hypermultiplets split into $17 \cdot 2 = 34$ localised and $273 - 34 = 239$ non-localised ones. All results of this section are summarized in table 6.3.

(μ_f, μ_g)	$(1,4)$	$(1,5)$	$(1,7)$
m	3	3	3
Enhancements	$\mathrm{III} \to \mathrm{I}_0^*$	$\mathrm{III} \to \mathrm{I}_0^*$	$\mathrm{III} \to \mathrm{I}_0^*$
Gauge Group	$SU(2)$	$SU(2)$	$SU(2)$
$n_V = \dim(G)$	3	3	3
$\mathrm{rk}(G)$	1	1	1
$h^{1,1}(\tilde{Y}_3)$	3	3	3
B_1	11	11	11
$\chi_{\text{top}}(X_{P_1})$	3	3	3
ϵ_1	7	11	19
a	2	3	5
m_P	1	2	4
$\chi_{\text{top}}(\tilde{Y}_3)$	-445	-434	-412
UnlocUnch Hypers n_{H_0}	221	210	188
LocUnch Hypers $n_{H_{LocUnch}}$	11	22	44
LocCh Hypers $n_{H_{ch}}$	44	44	44
Representation	2×fund.	2×fund.	2×fund.
$273 - (n_{H_0} + n_{H_{ch}} + n_{H_{LocUnch}} - n_V)$	0	0	0

Table 6.4.: Non-resolvable models with non-trivial gauge group. a characterises the form of the codimension-two singularity: $z^a + x_1^2 + x_2^2 + x_3^2 = 0$. m_P denotes the corresponding Milnor number.

6.4. Non-fully-resolvable Models with Non-trivial Gauge Group

Finally we would like to increase complexity a bit and consider models which are not resolvable and have a non-trivial gauge group. It is expected that at codimension-two loci both charged hypermultiplets (via M2-brane wrappings) and uncharged hypermultiplets (arising from the residual singularity) live.

We will analyse models with $\mathrm{III} \to \mathrm{I}_0^*$ enhancements. They are summarized in table 6.4.[1] To these models the same logic applies as to the previous ones except the fact that one has to take care of both charged and uncharged hypermultiplets located at codimension two. For details of the computation the reader is referred to appendix A.

Note that also the considerations about the gauge anomaly carry over. Again we have $G = SU(2)$ and $B_1 = 11$. This leads to the same prediction, namely two fundamental hypermultiplets located at each enhancement point (see section 6.1.5).

[1] Note that we skip the $(1,6)$-model. This is due to the fact that we do not know how to compute CxDef in this case since our formulae hold only for rational homology manifolds which is not the case here (see [AGW16]).

7. Models With Two Identical 7-branes

The analysis of F-theory models with one brane is generalized to models with two identical branes in generic position to each other, i.e. models with two identical gauge group factors. This provides a richer structure of singularity types and geometry of the fibration. First we generalize the relevant formulae to two identical branes. Then we apply them to several classes of models.
We will consider models with two identical branes wrapped on the divisors Σ_1 and Σ_2 which shall be described by the vanishing locus of the two variables: z_1 and z_2. Technically these models are defined via the requirement that f and g in the Weierstrass model vanish to certain orders along the divisors. Since the two branes shall be identical we need two numbers to specify a model. We employ the following notation: The model with $f = (z_1 z_2)^{\mu_f} f_0$ and $g = (z_1 z_2)^{\mu_g} g_0$ is called $[\mu_f \mu_g]$-*model.*

7.1. Calculation of χ_{top}

One essential ingredient of our considerations above was the topological Euler characteristic of the total (elliptically fibered) space Y_3 which contributes to the number of uncharged hypermultiplets. However formula (5.8) is specified to models in which the discriminant splits into two irreducible parts, Σ_0 and Σ_1. Our task now is it to generalize the computation to a threefold splitting of the discriminant: Σ_0, Σ_1 and Σ_2 where the fibers over Σ_1 and Σ_2 are identical.
Going back to the derivation, we started with four contributions to the topological Euler characteristic (see (5.4) to (5.7)). The generalization to two identical branes goes as follows.

- The vanishing orders of Δ along Σ_1 and Σ_2 are equal and denoted again by m.
- There are two types of codimension-two enhancement loci. First, there is one point where the branes intersect $\Sigma_1 \cap \Sigma_2$ which is referred to as R. Second, there are B loci in $\Sigma_0 \cap \Sigma_1$ and the same number in $\Sigma_0 \cap \Sigma_2$ according to the assumption that the situation is symmetric in $\Sigma_1 \leftrightarrow \Sigma_2$. The set of all points in the intersection of the residual discriminant and Σ_1 and Σ_2 is called P.
- There is an additional contribution to the topological Euler characteristic from the enhancement over R: $\chi_{top}(\pi^{-1}(R)) = 1 \cdot \chi_{top}(X_R)$.
- Contribution (5.5) which took the topology of the brane without all enhancement points into account becomes:

$$\begin{aligned}\chi_{top}\left(\pi^{-1}\big((\Sigma_1 \cup \Sigma_2) \setminus P \setminus R\big)\right) &= 2 \cdot \chi_{top}\big(\pi^{-1}(\Sigma_1 \setminus P_1 \setminus R)\big) \\ &= 2m \cdot \big(2 - 2g(\Sigma_1) - (B+1)\big).\end{aligned}$$

- Contribution (5.6) relies on the fact that $\Sigma_0 \in -12K_B - m\Sigma_1$ in the old situation. Here, $\Sigma_0 \in -12K_B - m\Sigma_1 - m\Sigma_2 = -12K_B - 2m\Sigma_1$. Therefore we have to replace $m \to 2m$.
- Similarly one has to replace $\mu_f \to 2\mu_f$ and $\mu_g \to 2\mu_g$ in the formula for the number of cusps.
- With the same argument as before the intersection multiplicity $\mu_i(f,g)$ can be set to zero.

Then the contributions to $\chi_{top}(Y_3)$ are:

- $\chi_{top}\big(\pi^{-1}(P)\big) = 2B \cdot \chi_{top}(X_P)$.
- $\chi_{top}\big(\pi^{-1}(R)\big) = \chi_{top}(X_R)$.
- $\chi_{top}\left(\pi^{-1}\big((\Sigma_1 \cup \Sigma_2) \setminus P \setminus R\big)\right) = 2m \cdot (2 - 2g - (B+1))$.
- $\chi_{top}\big(\pi^{-1}(\Sigma_0 \setminus Q \setminus P)\big) = -11 \cdot 12\, K_B^2 + 2m\, K_B \cdot \Sigma_1 + 4m^2\, \Sigma_1^2 + 4m\, \Sigma_1 \cdot \Sigma_0 + 2\,\epsilon B + C$.
- $\chi_{top}\big(\pi^{-1}(Q)\big) = 2\,C$ with the number of cusps $C = 24K_B^2 + \Big(8\mu_g + 12\mu_f\Big)\, K_B \cdot \Sigma_1 + 4\,\mu_f \mu_g\, \Sigma_1^2$.

All in all, χ_{top} is given by:

$$\begin{aligned}\chi_{\text{top}}(\tilde{Y}_3) = -540 + \chi_{\text{top}}(X_R) + 2\,B\,\Big(\chi_{\text{top}}(X_P) + \epsilon\Big) + m \cdot \Big(140 - 4\,m - 2B\Big)\\ -72\,\mu_g - 108\,\mu_f + 12\,\mu_f \mu_g.\end{aligned} \tag{7.1}$$

For the [00]-model, i.e. the general Weierstrass model the formula reduces to $\chi_{\text{top}}(\tilde{Y}_3) = -540$ which is the correct value.

7.2. The $[11]$-model

Let us start out with the [11]-model and work our way forward to more complicated models. By definition, $\mu_f = 1 = \mu_g$. The discriminant reads:

$$\Delta = \underbrace{z_1^2}_{\Sigma_1}\ \underbrace{z_2^2}_{\Sigma_2}\ \underbrace{(27g_0^2 + 4z_1 z_2\, f_0^3)}_{\Sigma_0}.$$

We read off $m = 2$. The fiber over Σ_1 and Σ_2 is type II (see table 4.1). So we do not have a gauge group in this model. When we approach the points where $g_0 = 0$ the fiber enhances to type III. Since $g_0 \in \mathcal{O}(18-2)$ there are $B = 16$ such points. Since the model is not resolvable we have to set $\chi_{\text{top}}(X_P) = 2 = \chi_{\text{top}}(X_R)$. Finally near a point P the residual discriminant Σ_0 takes the form $g_0^2 + z_i = 0$ if one sets all irrelevant prefactors to one. Because this equation defines a smooth curve, $\epsilon = -1$.
Plugging all these values into (7.1) we find $\chi_{\text{top}}(\tilde{Y}_3) = -474$. From equation (5.2) we obtain that there are $239 + \frac{1}{2}\sum m_P$ complex structure deformations. As in the one brane model with $\mu_f = 1 = \mu_g$ the local form of the singularity at all codimension-two enhancement loci is given by $z^3 + x_1^2 + x_2^2 + x_3^2 = 0$ which has Milnor number 2. Thus there are 272 complex structure deformations and 273 uncharged hypermultiplets. Since there are no gauge bosons present this spectrum has no gravitational anomalies.

7.3. The $[n1]$-models ($2 \leq n \leq 6$)

Here we follow the same line of thought. Since f, g vanish to order $n, 1$ along each brane, respectively, the discriminant is:

$$\Delta = z_1^2 z_2^2 \left(27 g_0^2 + 4\,(z_1 z_2)^{3n-2}\, f_0^3\right).$$

Therefore (f, g, Δ) vanish to orders $(n, 1, 2)$, i.e. $m = 2$, at codimension-one along each brane. This corresponds to fiber type II, i.e. there is no gauge group involved and the model cannot be resolved. There are two different enhancement loci. First, at $z_1,\ g_0 \to 0$ and $z_2,\ g_0 \to 0$ the vanishing orders increase to $(n, 2, 4)$ and second at $z_1, z_2 \to 0$ they increase to $(2n, 2, 4)$. Both correspond to fiber type IV. Since g_0 is in $\mathcal{O}(18-2)$ there are 32 points in the first class (B = 16) and one point where the two branes meet. Since the model is not resolvable we have to set $\chi_{\text{top}}(X_R) = 2 = \chi_{\text{top}}(X_{P_1})$. With these values (7.1) simplifies to:

$$\chi_{\text{top}}(\tilde{Y}_3) = -346 - 96\,n + 32\,\epsilon.$$

The remaining task is to determine ϵ. The curve Σ_0 takes the form $g_0^2 + z_{1,2}^{3n-2} = 0$ near an intersection locus with Σ_1 or Σ_2. Hence, $\epsilon = 3n - 4$ (see (5.10)). This cancels the n-dependence of $\chi_{\text{top}}(\tilde{Y}_3)$ and we end up with the value $\chi_{\text{top}}(\tilde{Y}_3) = -474$ for all $n \geq 2$. The local form of the singularity of both enhancement types is the same as in the previous model including the Milnor number of the singularities. Thus we find 273 uncharged hypermultiplets generated by the complex structure deformations.

Remark. One might wonder why the vanishing order of f is restricted to be smaller than 6 although all n dependency cancels. This is due to the fact that $f \in \mathcal{O}(12)$. For $n = 6$ the section f is completely fixed and does not have a generic part any more. Higher n are obviously not possible.

7.4. The $[1n]$-models ($n > 1$)

Let us outline the general structure of the models and then look at them in turn. The discriminant is given by:

$$\Delta = z_1^3 z_2^3 \left(4 f_0^3 + 27\,(z_1 z_2)^{2n-3}\, g_0^2\right). \tag{7.2}$$

The fiber over the branes is type III (vanishing orders $(f, g, \Delta) = (1, n, 3)$). Therefore the gauge group of the model is $G = SU(2) \times SU(2)$. At the intersection locus $\Sigma_1 \cap \Sigma_2$ we observe a III $\to$ I_0^* enhancement. The other enhancement locus ($\Sigma_0 \cap \Sigma_1$ and $\Sigma_0 \cap \Sigma_2$) has a type IV fiber for $n = 2$ and a type I_0^* fiber for $n \geq 3$. The integer B equals ten due to the vanishing orders of f along Σ_i.

With this information we are in the position to determine the matter spectrum completely employing the anomaly constraints for both the gravitational and the gauge anomalies (see equations (3.3) to (3.8)). Recall that the necessary group theoretic quantities A_R, B_R and C_R of $SU(2)$ are given in table 6.2.

- Constraint (3.4) is trivial.
- Constraint (3.5) is satisfied for $K_B = -3H$ and $n_T = 0$.
- From constraint (3.6) and (3.7) we can conclude that we need two fundamental representations at every $z_i = 0 = f_0$ locus and two matter fields in the fundamental at the brane intersection locus.
- From constraint (3.8) it is clear that we need one bifundamental representation at the brane intersection locus.
- From enhancement type III the theory receives gauge group $SU(2) \times SU(2)$ which comes with two times three gauge bosons ($n_V = 6$) and as always there are no tensor multiplets ($n_T = 0$) in the theory.
- All in all there must be four charged hypermultiplets at the brane intersection locus and four charged hypermultiplets at every other codimension-two locus (in total $n_{\text{charged}} = 4 + 20 \cdot 4 = 84$). This means that there must be 195 uncharged hypermultiplets either from the complex structure moduli or from localised uncharged singlets due to additional singularities.

In the following we would like to find this multiplet structure explicitly via resolving the models, computing $\chi_{\text{top}}(\tilde{Y}_3)$ and counting M2-brane wrapping possibilities. As we already did for the one brane models we would like to rewrite the models in Tate form and perform a toric resolution (see appendix B):

- In all models $m = 3$ and $B = 10$.
- Codimension-two enhancements: In model [12] there is a III $\rightarrow$ IV enhancement at $\Sigma_0 \cap \Sigma_i$. At $\Sigma_1 \cap \Sigma_2$ and at all codimension-two loci of the other models we observe a III $\rightarrow$ I_0^* enhancement. Note that locus $\Sigma_0 \cap \Sigma_i$ is still singular for the [14] and the [15] model. In most cases parts of the naïvely expected nodes in the fiber are deleted. The resulting χ_{top} is displayed in table 7.1.
- The gauge group is $SU(2) \times SU(2)$. Therefore, $n_V = 2 \cdot 3 = 6$ and $h^{1,1} = 2 + 2 = 4$.
- ϵ_1 is computed as usual (see section 5.3).
- At each $\Sigma_0 \cap \Sigma_i$ locus there live two fundamental representations of $SU(2)$ and at $\Sigma_0 \cap \Sigma_1$ there is a bifundamental of $SU(2) \times SU(2)$ located. In the appendix it is shown how these states can be interpreted in terms of M2-branes.
- The rest of the results are collected in table 7.1.

Having done the groundwork we can plug all values into (7.1) and compute the topological Euler characteristic of our fibrations (see figure 7.1). The details of the computation are completely analogous to the one brane case.

(μ_f, μ_g)	$(1,2)$	$(1,3)$	$(1,4)$	$(1,5)$
$\chi_{\text{top}}(X_{P_1})$	4	3	3	3
$\chi_{\text{top}}(X_R)$	4	4	4	4
ϵ_1	-1	3	7	11
a	–	–	2	3
m_P	0	0	1	2
$\chi_{\text{top}}(\tilde{Y}_3)$	-380	-380	-360	-340
UnlocUnch Hypers n_{H_0}	195	195	175	155
LocUnch Hypers $n_{H_{LocUnch}}$	–	–	20	40
LocCh Hypers $n_{H_{ch}}$	84	84	84	84
$273 - (n_{H_0} + n_{H_{ch}} + n_{H_{LocUnch}} - n_V)$	0	0	0	0

Table 7.1.: Models with non-trivial gauge group and two branes.

We observe that the gravitational anomaly condition vanishes in all cases. This confirms the mathematical fact that one has to take the residual singularities in terms of their Milnor numbers into account when computing the complex structure deformations. The whole picture remains consistent.

8. Conclusion and Outlook

To finish this work let us recapitulate our findings and main insights. We have studied F-theory on elliptically fibered Calabi-Yau threefolds with a special type of fibral singularities, type II fibers (in Kodaira's classification of singular elliptic fibers), at codimension-one which do not have a gauge group associated to them. Therefore it is not possible to resolve the singularity by introducing new fibral divisors which would lead to a gauge group with non-zero rank. Additionally we analysed models with type III fibers at codimension one, i.e. gauge group $SU(2)$, which showed singularities in the fiber even after performing the blow-up which is possible since there is a gauge group in this class of models.

It is known that all uncharged hypermultiplets are counted by the number of complex structure deformations which give rise to both non-localised and localised states. We have shown that if one includes the contribution from the terminal singularities in form of their Milnor number, the models are consistent, i.e. the complex structure deformations contributed exactly the correct number of hypermultiplets to cancel the gravitational anomaly.

We have addressed the question how the complex structure moduli are distributed into localised and non-localised hypermultiplets. The problem can be faced from two sides. First, the analysis of the I_1 conifold model of which we know that there lives one hypermultiplet at each singularity showed that the number of localised hypers coincides with the Milnor number. Second, we found strong mathematical arguments that the number of versal complex deformations, i.e. deformations which deform the singularity, is given by the Milnor number. The rest, i.e. $\text{CxDef}(Y_3) - \sum_P m_P$, are the unlocalised uncharged hypermultiplets. This gives an intuitive interpretation of the situation and explains why the total number of hypermultiplets is distributed in this fashion.

There are many ways to continue the path of this work. The most obvious idea is to consider other models which have a richer singularity structure and work out the details in order to show that the above assertion can be applied as well. Second, one could add an interpretation of the singularity in terms of M-theory degrees of freedom. In mathematical terms one would have to perform a non-flat resolution and analyse the fiber. Additionally the utilized mathematical theorems hold only for rational homology manifolds. It is not clear how to compute the number of complex structure deformations for more general manifolds. Moreover one could simply count the dimension of the vector space of homogeneous polynomials f_0 and g_0 which should equal the number of deformations which leaves the singularity invariant. However one has to be aware of the difference between toric and non-toric deformations. Therefore the naïve counting gives the wrong answer. Finally it would be worth to transfer our calculations to compactifications to four dimensions, i.e. compute the number of complex structure moduli and versal deformations of a Calabi-Yau fourfold. This is very challenging since the anomaly constraints in six dimensions are by far more informative than in four dimensions.

A. Toric Resolution of Models With One Brane

We want to add some interpretation to the analysis in the main text. There must be located (amongst others) charged matter representations at the codimension-two loci. This fact will be reproduced in terms of M-theory degrees of freedom, i.e. M2-branes. The blow-up of a F-theory model corresponds to moving on the Coulomb branch. In other words one breaks the gauge group to its Cartan subgroup. Then one can observe how the M2-brane wraps various linear combinations of the resolution divisor. This illustrates in a very nice way how the adjoint representation at the codimension-one locus, i.e. the 7-brane, and the matter representations at the codimension-two loci come into play.
To do so the prescription is:

- Write the model of consideration in Tate form.
- Perform the toric resolution and find the corresponding Stanley-Reisner ideal.
- Compute the singular locus of the resolved model and calculate its associated Milnor number.
- Analyse the codimension-one and codimension-two fibers in terms of their weights under the Cartan subgroup of the gauge group and find the representations: At codimension one an adjoint representation and at codimension two matter representations.

This procedure can only be applied to models with non-trivial non-abelian gauge group since one introduces new fibral divisor classes during the blow-up. The number of new divisors equals the rank of the gauge group, i.e. we cannot perform a blow-up along a divisor in the base if there is no gauge group present or in other words there is no Coulomb branch along we could move.

A.1. Models in Tate Form

We want to rewrite the Weierstrass models presented in the main text in Tate form: We define Tate models which reproduce the correct vanishing orders of f and g. Then the Tate model is resolved by introducing resolution divisors. The advantage of this formulation is that the resolution can be described in terms of toric geometry which we will desperately need.
The Tate form of an elliptic curve is given by:

$$y^2 + a_1\, xyz + a_3\, yz^3 = x^3 + a_2\, x^2z^2 + a_4\, xz^4 + a_6\, z^6, \tag{A.1}$$

where the a_n are sections of $\mathcal{O}(3n)$. In contrast recall that an elliptic curve in Weierstrass form is given by:

$$y^2 = x^3 + fx^2z^4 + gz^6,$$

(μ_f, μ_g)	a_1	a_2	a_3	a_4	a_6
$(1,2)$	1	1	1	1	2
$(1,3)$	1	2	2	1	3
$(1,4)$	2	3	2	1	4
$(1,5)$	2	4	3	1	5
$(1,6)$	3	5	3	1	6
$(1,7)$	3	6	4	1	7
$(2,2)$	1	1	1	2	3

Table A.1.: Vanishing orders of the sections a_i along the locus Σ_1. The gauge group is $SU(2)$ except for the last model which has gauge group $SU(3)$.

where f, g are sections of $\mathcal{O}(12), \mathcal{O}(18)$, respectively. It can be shown that the parameters of the Tate form and the Weierstrass form are related in the following fashion:

$$f = -\tfrac{1}{48}(b_2^2 - 24\, b_4), \quad g = -\tfrac{1}{864}(-b_2^3 + 36\, b_2 b_4 - 216\, b_6),$$

where the b_n are sections of $\mathcal{O}(3n)$. They take the form:

$$b_2 = a_1^2 + 4a_2, \quad b_4 = a_1 a_3 + 2a_4, \quad b_6 = a_3^2 + 4a_6.$$

In order to reproduce e.g. vanishing orders $\mu_f = 1, \mu_g = 2$ one has to set:

$$\begin{aligned} a_i &= z_1 \tilde{a}_i \quad \text{for } i \in \{1,2,3,4\}, \\ a_6 &= z_1^2 \tilde{a}_6. \end{aligned}$$

This has to be done for all models we considered in the main text. All choices for the vanishing orders of the a_i for all models are collected in table A.1. Having fixed the vanishing orders relabel $\tilde{a}_i \to a_i$ and keep in mind that the new a_i are not the full Tate sections.

A.2. Toric Blow-up and Stanley-Reisner Ideal

The next step is to perform the blow-up and compute the associated Stanley-Reisner ideal. In our models there occur exactly two different gauge groups: $SU(2)$ and $SU(3)$. For $SU(2)$ we have to introduce one additional divisor and for $SU(3)$ the number of Kähler moduli has to increase by two according to the rank of the two groups.

Since the models are singular at $x = y = z_1 = 0$ the correct way to do the blow-up in the $SU(2)$-case is:

$$x \to e_1 x, \quad y \to e_1 y, \quad z_1 \to e_0 e_1.$$

	x	y	z	e_0	e_1
$[Z]$	2	3	1	·	·
$[E_1]$	-1	-1	·	-1	1

(a) Fiber type III ($SU(2)$)

	x	y	z	e_0	e_1	e_2
$[Z]$	2	3	1	·	·	·
$[E_1]$	-1	-1	·	-1	1	·
$[E_2]$	-1	-2	·	-1	·	1

(b) Fiber type IV ($SU(3)$)

Table A.2.: Toric weights for the fiber ambient space.

For $SU(3)$ one has to replace:

$$x \to e_1 e_2 x, \quad y \to e_1 e_2^2 y, \quad z_1 \to e_0 e_1 e_2.$$

After doing so the hypersurface equation (A.1) factorises into powers of e_1 and e_2 and the proper transform which we denote by PT. It can be viewed as a hypersurface in a toric fiber ambient space with coordinates $x, y, z, e_0, e_1(, e_2)$ and toric weights displayed in table A.2. However the hypersurface as such is not the most general hypersurface compatible with the scaling relations table B.1. This most generic hypersurface would rather give rise to Kodaira fibers of type I_2, and not type III. In particular the dual polytope does not reproduce the monomial in PT. In this sense this type III model cannot be analysed via the technology of tops [CPR97] [BS03].
However we can still compute the Stanley-Reisner ideal (SRI) of the toric ambient space and analyse the hypersurface PT by hand. With the help of SageMath (see appendix C) we find as possible SRIs:

$$SU(2) : \langle ze_1, xyz, xye_0 \rangle,$$
$$SU(3) : \langle ye_1, ze_1, ze_2, xyz, xye_0, xe_0e_2 \rangle.$$

A.3. Residual Singular Locus and Associated Milnor Number

Most of the would-be singular locus of our models, i.e. critical points of the hypersurface equation $PT = 0$, are excluded by the SRIs. However there remain some singularities which are displayed in table A.3. Note that in the $(1, 3)$-model the codimension of the singular locus is too high and therefore non-existent: The equation $e_1 = 0$ fixes a curve in the base; $a_6 = 0$ determines a point on this curve; therefore there is no codimension left to impose $a_4 = 0$. These singularities are responsible for uncharged localised hypers (see main text).

A.4. Analysis of Codimension-One Loci

Now we would like to make explicit how the M-theory M2-brane wraps the different curves in the resolved fiber which gives rise to massless representations in the F-theory limit.
At codimension-one, i.e. along the 7-brane, we expect one adjoint representation because the gauge bosons live here.

(μ_f, μ_g)	Singular locus after resolution	a
$(1,2)$	$\varnothing$	–
$(1,3)$	$\{e_1\} \cap \{a_6\} \cap \{a_4\} \cap \{x\} \cap \{y\} = \varnothing$	–
$(1,4)$	$\{e_1\} \cap \{a_4\} \cap \{x\} \cap \{y\}$	3
$(1,5)$	$\{e_1\} \cap \{a_4\} \cap \{x\} \cap \{y\}$	3
$(1,6)$	$\{e_1\} \cap \{a_4\} \cap \{x\} \cap \{y\}$	4
$(1,7)$	$\{e_1\} \cap \{a_4\} \cap \{x\} \cap \{y\}$	5
$(2,2)$	$\varnothing$	–

Table A.3.: Singular locus of proper transforms. The singularity parameter a describes the local form of the singularity: $z^a + x_1^2 + x_2^2 + x_3^2$.

For definiteness let us focus on the $SU(2)$ case. Afterwards we will briefly present how the analysis has to be modified if there are more resolution divisors present. So let us analyse the singular fiber over $z_1 = 0$. The resolution divisors $e_0 = 0$ and $e_1 = 0$ are rational fibrations over the locus $z_1 = 0$ in the base, and $\pi^{-1}(z_0) = e_0 e_1$. The fiber over $z_1 = 0$ thus consists of two rational curves, given by the vanishing of e_0 and e_1, respectively. Concretely,

$$\mathbb{P}^1_A \,:\, PT|_{e_0 \to 0} = y^2 - e_1 x^3,$$

$$\mathbb{P}^1_B \,:\, PT|_{e_1 \to 0} = \begin{cases} y^2 - a_4 e_0\, x z^4 + a_3 e_0\, y z^3 - a_6 e_0^2\, z^6 & (1,2)\text{-model} \\ y^2 - a_4 e_0\, x z^4 & (1,i)\text{-models for } i = 3, \ldots, 7. \end{cases}$$

All equations do not factorize and therefore define two $\mathbb{P}^1$s in the fiber called $\mathbb{P}^1_A$ and $\mathbb{P}^1_B$. They intersect at $\{e_0\} \cap \{e_1\} \cap \{y\}$ with order two. We explicitly see a realisation of a type III fiber. Note that $\mathbb{P}^1_A$ is intersected by the zero-section of the Weierstrass model. Therefore the massless states we are interested in must not involve $\mathbb{P}^1_A$ because states wrapping $\mathbb{P}^1_A$ would be charged under the Kaluza-Klein $U(1)$, i.e. such states only give rise to KK modes as opposed to new zero modes by themselves. It therefore suffices to consider the remaining $\mathbb{P}^1$s.

Next we dive into the representation theory of $SU(2)$. Its simple root is $\alpha = -2$. Therefore the weight vector of the adjoint representation is $(-2, 0, 2)^T$. This is what we expected to find from the physics perspective. The charge of $\mathbb{P}^1_B$ under the Cartan of $SU(2)$ is given by the intersection product of the curve $\mathbb{P}^1_B$ and the divisor E_1 locally defined by $e_1 = 0$.

$$\mathbb{P}^1_B \circ E_1 = [e_1] \cdot \left[y^2 + \ldots\right] \cdot [e_1] = -[e_1] \cdot [y^2] \cdot [e_0] = -2,$$

where we used that $[e_1] = -[e_0]$ (see table A.2). Thus if a M2-brane wraps $\mathbb{P}^1_B$ we obtain a state with Cartan charge -2. An Anti-M2-brane can wrap $\mathbb{P}^1_B$ as well inducing a state of Cartan charge 2. The state uncharged under the Cartan comes from reduction of the M-theory three-from along $\mathbb{P}^1_B$. In this sense we understand how the adjoint representation over a codimension-one locus with associated gauge group comes about.

The same analysis has to be repeated in our $SU(3)$ case. Here the fibral rational curves are given by:

$$\begin{aligned}
\mathbb{P}^1_A &: \ PT|_{e_0\to 0} = y^2 e_2 - x^3 e_1 , \\
\mathbb{P}^1_B &: \ PT|_{e_1\to 0} = y\cdot (z^3 a_3 e_0 + y e_2) \\
\mathbb{P}^1_C &: \ PT|_{e_2\to 0} = a_3 e_0\, y z^3 + x^3 e_1 - a_6 e_0^3 e_1\, z^6 - a_4 e_0^2 e_1\, x z^4 - a_2 e_0 e_1\, x^2 z^2
\end{aligned}$$

It seems that the second equation factorises. However, the locus $\{e_1\}\cap\{y\}$ is forbidden by the SRI. The remaining three curves intersect in one point: $\{e_0\}\cap\{e_1\}\cap\{e_2\}$ which is as we expected a type IV fiber.

A.5. Analysis of Codimension-Two Loci

The next step is to take a look at the fiber enhancement at codimension two. The discriminant in our cases takes the form:

$$\Delta = \begin{cases} \frac{1}{16}\, z_1^3 \cdot \left(64\, a_4^3 + \mathcal{O}(z_1)\right), & (1,i)\text{-models}, \\ \frac{1}{16}\, z_1^4 \cdot \left(27\, a_3^4 + \mathcal{O}(z_1)\right), & (2,2)\text{-model}. \end{cases}$$

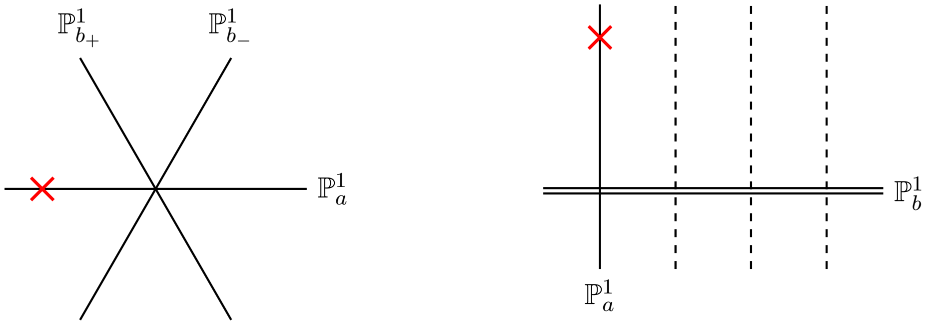
One can read off that the fiber enhances at $a_4 \to 0$, $a_3 \to 0$, respectively. As before we concentrate on the $SU(2)$ cases first and treat the $SU(3)$ model afterwards. In the $(1,2)$-model we find:

$$\begin{aligned}
PT|_{e_0, a_4\to 0} &= \underbrace{y^2 - e_1 x^3}_{\mathbb{P}^1_A}, \\
PT|_{e_1, a_4\to 0} &= \underbrace{y^2 + a_3 e_0\, y z^3 - a_6 e_0^2\, z^6}_{\mathbb{P}^1_B \text{ at } a_4\to 0} \\
&= \underbrace{\left(y + \tfrac{1}{2} a_3 e_0\, z^3 + \sqrt{a_3^2 + 4a_6}\, e_0\, z^3\right)}_{\mathbb{P}^1_{b_+}} \cdot \underbrace{\left(y + \tfrac{1}{2}\, a_3 e_0\, z^3 - \sqrt{a_3^2 + 4a_6}\, e_0\, z^3\right)}_{\mathbb{P}^1_{b_-}} .
\end{aligned}$$

The vanishing of a_4 enables us to factorize the equation for $\mathbb{P}^1_B$. In other words the curve $\mathbb{P}^1_B$ splits into two curves, called $\mathbb{P}^1_{b_+}$ and $\mathbb{P}^1_{b_-}$, at the codimension-two enhancement locus. All three curves meet at $\{e_0\}\cap\{e_1\}\cap\{y\}$ with multiplicity one. In Kodaira-Tate language this is called Type IV which was expected by the vanishing orders of f, g and Δ (see figure A.1). For the $(1,i)$-models with $i > 2$ the situation is slightly different:

$$\begin{aligned}
\mathbb{P}^1_a &: \ PT|_{e_0, a_4\to 0} = y^2 - e_1 x^3 , \\
\mathbb{P}^1_b &: \ PT|_{e_1, a_4\to 0} = y^2 .
\end{aligned}$$

The second equation does not factorise any more. It rather describes a non-reduced curve of multiplicity two. However, we will see below that the M2-brane can wrap the curve $\mathbb{P}^1_b$ twice. Therefore the situation for all $(1,i)$-models with $i \geq 2$ is the same.



(a) $(1, 2)$-model (b) $(1, i)$-models, $i > 2$

Figure A.1.: Affine Dynkin diagram of the resolved $\{z_1\} \cap \{a_4\}$ locus. The red cross denotes the intersection with the zero-section $z = 0$ of the Weierstrass model.

Next we need the Cartan weights of $\mathbb{P}^1_{b_\pm}$ and $\mathbb{P}^1_b$. $\mathbb{P}^1_a$ is still intersected by the zero section and does not play a role in our analysis. Since the equations for the curves are symmetric and since the weights of have to add up to the weight of $\mathbb{P}^1_B$ we expect it to be -1. Nevertheless let us check it explicitly:

$$\begin{aligned}\mathbb{P}^1_b \circ E_1 &= [e_1] \cdot [y^2] \cdot [e_1] = -2 \cdot [e_0] \cdot [y] \cdot [e_1] = -2,\\ \mathbb{P}^1_{b_\pm} \circ E_1 &= [e_1] \cdot \left[y + \tfrac{1}{2} a_3 e_0 \, z^3 \pm \sqrt{a_3^2 + 4a_6 \, e_0 \, z^3} \right] \cdot [e_1] = -[e_1] \cdot [y] \cdot [e_0] = -1.\end{aligned}$$

If we assume that the curve $\mathbb{P}^1_b$ can be wrapped twice the situation is completely equivalent in both cases. The $SU(2)$ root is given by $\mathbb{P}^1_B$ which has factorized into $\mathbb{P}^1_{b_+} + \mathbb{P}^1_{b_-}$ in the $(1, 2)$-case. The highest weight of the fundamental representation of $SU(2)$ is $w = -1$ which is either represented by $\mathbb{P}^1_{b_+}$ or $\mathbb{P}^1_{b_-}$ (or by one wrapping of "$\frac{1}{2}\,\mathbb{P}^1_b$"). In this fashion we can build:

$$\begin{aligned} w = \mathbb{P}^1_{b_+} :& \quad \begin{pmatrix} w - \alpha \\ w \end{pmatrix} = \begin{pmatrix} \mathbb{P}^1_{b_+} - (\mathbb{P}^1_{b_+} + \mathbb{P}^1_{b_-}) \\ \mathbb{P}^1_{b_+} \end{pmatrix} = \begin{pmatrix} -\mathbb{P}^1_{b_-} \\ \mathbb{P}^1_{b_+} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \\ w = \mathbb{P}^1_{b_-} :& \quad \begin{pmatrix} w - \alpha \\ w \end{pmatrix} = \begin{pmatrix} \mathbb{P}^1_{b_-} - (\mathbb{P}^1_{b_+} + \mathbb{P}^1_{b_-}) \\ \mathbb{P}^1_{b_-} \end{pmatrix} = \begin{pmatrix} -\mathbb{P}^1_{b_+} \\ \mathbb{P}^1_{b_-} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{aligned}$$

At first sight it looks more that we only get one **2** of SU(2) and its conjugate because the two sets of curves are just related by a minus sign. However fom the gravitational anomalies we know that there must be located two fundamental representations at one locus (see table 6.1 and table 6.4). A potential explanation for this phenomenon is that the fundamental representation of $SU(2)$ has a speciality: it is real, i.e. it is isomorphic to its conjugate. Therefore one presumably has to count the above to wrappings independently.

Now let us turn to the $SU(3)$ case. Here the enhancement happens at $a_3 = 0 = z_1$ and the fiber takes

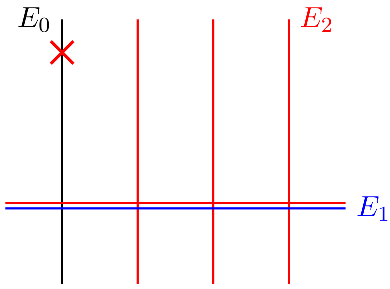


Figure A.2.: Affine Dynkin diagram of the resolved $\{z_1\} \cap \{a_3\}$ locus. The red cross denotes the intersection with the zero-section $z = 0$ of the Weierstrass model. The blue and red colour indicate how the $\mathbb{P}^1$s of the codimension-one fiber split into the ones of the codimension-two fiber.

the form:

$$
\begin{aligned}
PT|_{e_0 \to 0,\, a_4 \to 0} &= y^2 e_2 - x^3 e_1, \\
PT|_{e_1 \to 0,\, a_4 \to 0} &= y^2 e_2, \\
PT|_{e_2 \to 0,\, a_4 \to 0} &= -e_1 \cdot (x^3 + a_2 x^2 (e_0 z^2) + a_4 x (e_0 z^2)^2 + a_6 (e_0 z^2)^3) \\
&= e_1 \cdot (x - e_0 z^2 \cdot f_1(a_i)) \cdot (x - e_0 z^2 \cdot f_2(a_i)) \cdot (x - e_0 z^2 \cdot f_3(a_i)).
\end{aligned}
$$

The last equation factors four times. Therefore we observe six components intersecting as shown in figure A.2.[1] This is a type I_0^* fiber as expected. Let us denote the black curve in the figure by $\mathbb{P}^1_a$, the double curve by $\mathbb{P}^1_b$ and the other red curves by $\mathbb{P}^1_c, \mathbb{P}^1_d$ and $\mathbb{P}^1_e$.
$SU(3)$ has two simple roots:

$$\alpha_1 = (-2, 1), \quad \alpha_2 = (1, -2).$$

These two roots must be represented by $\mathbb{P}^1_b$, i.e. all blue curves, and $\mathbb{P}^1_b + \mathbb{P}^1_c + \mathbb{P}^1_d + \mathbb{P}^1_e$, i.e. all red curves (the colours refer to figure A.2). Let us check this. The intersection numbers are given by:

- $\mathbb{P}^1_a \circ E_1 = [a_3] \cdot [e_1] \cdot [e_0] \cdot [y^2 e_2 - \cancel{x^3 e_1}] = 1$ since y cannot vanish (ye_1 is in the SRI).
- $\mathbb{P}^1_a \circ E_2 = [a_3] \cdot [e_2] \cdot [e_0] \cdot [\cancel{y^2 e_2} - x^3 e_1] = 1$ since x cannot vanish (xe_0e_2 is in the SRI).
- $\mathbb{P}^1_b \circ E_1 = [a_3] \cdot [e_1] \cdot [e_1] \cdot [e_2]$. Use $[e_1] = [y] - 2[x]$ and ye_1 is in the SRI. Then, $\mathbb{P}^1_b \circ E_1 = [a_3] \cdot [e_1] \cdot (\cancel{[y]} - 2[x]) \cdot [e_2] = -2$.
- $\mathbb{P}^1_b \circ E_2 = [a_3] \cdot [e_2] \cdot [e_1] \cdot [e_2]$. Use $[e_2] = [x] - [y]$ and ye_1 is in the SRI. Then, $\mathbb{P}^1_b \circ E_2 = [a_3] \cdot [e_2] \cdot [e_1] \cdot ([x] - \cancel{[y]}) = 1$.
- $\mathbb{P}^1_{c,d,e} \circ E_1 = [a_3] \cdot [e_1] \cdot [e_2] \cdot [x - f_i(a_i) e_0 z^2] = 1$.

[1] By the way: The intersecting pattern can be computed by `SageMath` as well. One has to define the ideals associated to the $\mathbb{P}^1$s in Singular, add them in pairs and perform a primary decomposition which returns the irreducible components of the intersection locus.

- $\mathbb{P}^1_{c,d,e} \circ E_2 = [a_3] \cdot [e_2] \cdot [e_2] \cdot [x - f_i(a_i)e_0z^2]$. Use $[e_2] = -[e_1] - [e_0]$ and xe_0e_2 is in the SRI. Then, $\mathbb{P}^1_{c,d,e} \circ E_2 = [a_3] \cdot [e_2] \cdot (-[e_1] - \cancel{[e_0]}) \cdot [x - f_i(a_i)e_0z^2] = -1$.

All in all,

$$\mathbb{P}^1_a : (1,1), \quad \mathbb{P}^1_b : (-2,1), \quad \mathbb{P}^1_{c,d,e} : (1,-1),$$

where the ith entry in the vector denotes the charges under E_i. One simple root must be associated to the sum of all blue curves and the other to the sum of all red curves. Let us check this:

$$\begin{aligned} \mathbb{P}^1_b : (-2,1) \quad &\longleftrightarrow \quad \alpha_1 \quad \checkmark \\ \mathbb{P}^1_b + \mathbb{P}^1_c + \mathbb{P}^1_d + \mathbb{P}^1_e : (-2,1) + 3 \cdot (1,-1) = (1,-2) \quad &\longleftrightarrow \quad \alpha_2 \quad \checkmark \end{aligned}$$

The highest weight of the fundamental representation is $w_1 = (1,0)$. There are three possibilities to represent the highest weight in terms of curve wrappings:

$$-(\mathbb{P}^1_b + \mathbb{P}^1_c) : (1,0), \quad -(\mathbb{P}^1_b + \mathbb{P}^1_d) : (1,0), \quad -(\mathbb{P}^1_b + \mathbb{P}^1_e) : (1,0).$$

To construct the other states, we have to act with the simple roots on the highest weight vector. Let us consider the first of the above three wrapping possibilities.

$$\begin{aligned} w_1 :& - (\mathbb{P}^1_b + \mathbb{P}^1_c), \\ w_1 + \alpha_1 :& - (\mathbb{P}^1_b + \mathbb{P}^1_c) + \mathbb{P}^1_b = -\mathbb{P}^1_c, \\ w_1 + \alpha_1 + \alpha_2 :& - \mathbb{P}^1_c + \mathbb{P}^1_b + \mathbb{P}^1_c + \mathbb{P}^1_d + \mathbb{P}^1_e = \mathbb{P}^1_b + \mathbb{P}^1_d + \mathbb{P}^1_e. \end{aligned}$$

Note, that no multiple wrappings appear, all wrappings are either with positive or with negative orientation, and all combinations of $\mathbb{P}^1$s are connected (see figure A.2). We conclude that we find three copies of the fundamental representation in the resolved fiber which is what we need to cancel the anomalies (see table 6.4).

A.6. Wrap-up

We have considered Tate models with III $\to$ IV and IV $\to$ I_0^* enhancements, have resolved them and have shown that the structure of the resolution is such that we find two fundamental representations of the gauge group $SU(2)$ and three fundamentals in the $SU(3)$ model at each codimension-two enhancement locus. Our findings are strengthened by the results of [GM00]. Here the same Tate models are analysed in terms of their gravitational anomalies and the result that these representations live at each codimension-two locus coincides. We added an interpretation in the context of M-theory and M2-brane wrappings and showed that everything nicely fits together.

A.7. SageMath Code

Finally, I would like to share the `SageMath` code with the reader which is an essential part of the above analysis.

```
singular.lib('primdec.lib')
singular.lib('sing.lib')

R = singular.ring(0, '(x,y,z,a1,a2,a3,a4,a6,z1,e0,e1,e2)', 'dp')
x, y, z, a1, a2, a3, a4, a6, z1, e0, e1, e2 = var('x,y,z,a1,a2,a3,a4,a6,z1,e0,e1,e2')

def displaySingularLocus(ideal):
   sL = singular.slocus(ideal.std()).std().minAssGTZ()
   for i in range(1, len(sL) + 1):
      sL[i] = sL[i].std()
   print str(sL) + "“n"

def myAnalysis(n1,n2,n3,n4,n6,gaugeGroup = "",output = 1):
   """
   Performs the Blow-up, computes the residual singular locus and returns the proper transform.

   n1,...,n6: vanishing orders of a1,...,a6 in the Tate model

   gaugeGroup: "SU(2)" or "SU(3)" or "". Is needed to perform the correct blow-up

   output: controls whether any print commands display intermediate steps
   """
   if output:
      print "--------------------------------------------------------------"
      print "Analysing model with ("+str(n1)+","+str(n2)+","+str(n3)+","+str(n4)+","+str(n6)+")"
      print "--------------------------------------------------------------"

   Tate = y2 + x*y*z*a1 + y*z3*a3 - (x3 + x2*z2*a2 + x*z4*a4 + z6*a6)
   p = Tate.substitute(a1=z1n1*a1).substitute(a2=z1n2*a2).substitute(a3=z1n3*a3).substitute(a4=z1n4*a4).substitute(a6=z1n6*a6)

   if output:
      print "Singular locus of Tate form:"
      displaySingularLocus(singular.ideal(str(p)))

   if(gaugeGroup == "SU(2)"):
      p1 = p.substitute(x=e1*x).substitute(y=e1*y).substitute(z1=e0*e1)
      if output: print "A possible SRI is: z*e1, x*y*z, x*y*e0."
   elif (gaugeGroup == "SU(3)"):
      p1 = p.substitute(x=e1*e2*x).substitute(y=e1*e22*y).substitute(z1=e0*e1*e2)
```

```
        if output: print ”A possible SRI is: y*e1, z*e1, z*e2, x*y*z, x*y*e0, x*e0*e2.”
    else:
        print ”The Tate form is given by: ” + str(p)
        return True #This is the end of the analysis since a resolution is not possible.

    # Remove exceptional divisors from equation
    PT = singular.ideal(str(p1)).std().minAssGTZ()[2].std()
    if output:
        print ”“nProper transform:“n” + str(PT)

        print ’“nSingular Locus of proper transform:’
        displaySingularLocus(PT)

        print ”“n“n”
    return PT

def analyseCodim1(PT,gaugeGroup = ””):
    print ”ANALYSIS OF CODIMENSION-ONE LOCUS”
    if(gaugeGroup == ”SU(2)”):
        E0 = (singular.ideal(’e0’).std() + PT).std()
        E1 = (singular.ideal(’e1’).std() + PT).std()
        print ’E0:“n’ + str(E0.minAssGTZ())
        print ’E1:“n’ + str(E1.minAssGTZ())
    elif(gaugeGroup == ”SU(3)”):
        E0 = (singular.ideal(’e0’).std() + PT).std()
        E1 = (singular.ideal(’e1’).std() + PT).std()
        E2 = (singular.ideal(’e2’).std() + PT).std()
        print ’E0:“n’ + str(E0.minAssGTZ())
        print ’E1:“n’ + str(E1.minAssGTZ())
        print ’E2:“n’ + str(E2.minAssGTZ())
    else:
        print ”ERROR”
        return False
    print ”“n”

def analyseCodim2(PT,gaugeGroup = ””):
    print ”ANALYSIS OF CODIMENSION-TWO LOCUS”
    if(gaugeGroup == ”SU(2)”):
        print ”Analyse a4 --¿ 0.”
        E0 = (singular.ideal(’e0’).std() + PT + singular.ideal(’a4’).std()).std()
        E1 = (singular.ideal(’e1’).std() + PT + singular.ideal(’a4’).std()).std()
        print ’E0:“n’ + str(E0.minAssGTZ())
        print ’E1:“n’ + str(E1.minAssGTZ())
    elif(gaugeGroup == ”SU(3)”):
        print ”Analyse a3 --¿ 0.”
```

```
        E0 = (singular.ideal('e0').std() + PT + singular.ideal('a3').std()).std
            ()
        E1 = (singular.ideal('e1').std() + PT + singular.ideal('a3').std()).std
            ()
        E2 = (singular.ideal('e2').std() + PT + singular.ideal('a3').std()).std
            ()
        print 'E0:“n' + str(E0.minAssGTZ())
        print 'E1:“n' + str(E1.minAssGTZ())
        print 'E2:“n' + str(E2.minAssGTZ())
    else:
        print "ERROR"
        return False
    print "“n"


PT12 = myAnalysis(1,1,1,1,2,"SU(2)",0)
PT13 = myAnalysis(1,2,2,1,3,"SU(2)",0)
PT14 = myAnalysis(2,3,2,1,4,"SU(2)",0)
PT15 = myAnalysis(2,4,3,1,5,"SU(2)",0)
PT16 = myAnalysis(3,5,3,1,6,"SU(2)",0)
PT17 = myAnalysis(3,6,4,1,7,"SU(2)",0)
PT22 = myAnalysis(1,1,1,2,3,"SU(3)",0)

analyseCodim1(PT12,"SU(2)")
analyseCodim1(PT13,"SU(2)")
analyseCodim1(PT14,"SU(2)")
analyseCodim1(PT15,"SU(2)")
analyseCodim1(PT16,"SU(2)")
analyseCodim1(PT17,"SU(2)")
analyseCodim1(PT22,"SU(3)")

analyseCodim2(PT12,"SU(2)")
analyseCodim2(PT13,"SU(2)")
analyseCodim2(PT14,"SU(2)")
analyseCodim2(PT15,"SU(2)")
analyseCodim2(PT16,"SU(2)")
analyseCodim2(PT17,"SU(2)")
analyseCodim2(PT22,"SU(3)")
```

B. Toric Resolution of Models With Two Branes

The procedure of resolving these models is very similar to the above ones with one brane. Therefore the calculations are only sketched in this chapter. It turns out that one has to distinguish the cases for different values of n. We will consider $n = 2, 3, 4, 5$ since all conceptional features appear already here.

First note that the vanishing orders of the sections a_i in the Tate model are chosen exactly as in the one brane case (see table A.1). As an example, the model [14] is defined by:

$$a_1 = (z_1 z_2)^2\, \tilde{a}_1, \qquad a_2 = (z_1 z_2)^3\, \tilde{a}_2, \qquad a_3 = (z_1 z_2)^2\, \tilde{a}_3,$$
$$a_4 = z_1 z_2\, \tilde{a}_4, \qquad a_6 = (z_1 z_2)^4\, \tilde{a}_1.$$

As before relabel $\tilde{a}_i \to a_i$ to simplify the notation. This leads to the correct vanishing behaviour of (f, g, Δ) such as to reproduce type III singularities along z_1 and z_2. In particular,

$$\Delta = \tfrac{1}{16} z_1^3 z_2^3 \left(64\, a_4^3 + \mathcal{O}(z_1, z_2)\right). \tag{B.1}$$

The singular locus of the so-defined Tate form is as expected $\{x\} \cap \{y\} \cap \{z_1\}$ and $\{x\} \cap \{y\} \cap \{z_2\}$. Now let us blow up the model by replacing:

$$x \to e_1 f_1 x, \quad y \to e_1 f_1 y, \quad z_1 \to e_0 e_1, \quad z_2 \to f_0 f_1. \tag{B.2}$$

The proper transform PT of the Tate form after the blow-up is given by:

$$PT = -e_1 f_1\, x^3 + y^2 + a_1\, e_0^n e_1^n f_0^n f_1^n\, xyz - a_2\, e_0^n e_1^n f_0^n f_1^n\, x^2 z^2 + a_3\, e_0^n e_1^{n-1} f_0^n f_1^{n-1}\, yz^3$$
$$-a_4\, e_0 f_0\, xz^4 - a_6\, e_0^n e_1^{n-2} f_0^n f_1^{n-2}\, z^6 \quad \text{for } n \geq 2.$$

Again we view PT as a hypersurface in a toric fiber ambient space with coordinates x, y, z, e_1, f_1, e_0, f_0. The associated toric weights are displayed in table B.1. One possible choice for the Stanley-Reisner

	x	y	z	e_0	e_1	f_0	f_1
$[Z]$	2	3	1	·	·	·	·
$[E_1]$	−1	−1	·	−1	1	·	·
$[F_1]$	−1	−1	·	·	·	−1	1

Table B.1.: Scaling Relations for the fiber ambient space.

Model	Singular locus after resolution	a
[12]	$\varnothing$	–
[13]	$\varnothing$	–
[14]	$\big(\{e_1\} \cap \{a_4\} \cap \{x\} \cap \{y\}\big) \cup \big(\{f_1\} \cap \{a_4\} \cap \{x\} \cap \{y\}\big)$	3
[15]	$\big(\{e_1\} \cap \{a_4\} \cap \{x\} \cap \{y\}\big) \cup \big(\{f_1\} \cap \{a_4\} \cap \{x\} \cap \{y\}\big)$	3

Table B.2.: Singular locus of proper transforms. The singularity parameter a describes the local form of the singularity: $z^a + x_1^2 + x_2^2 + x_3^2$.

ideal is (see appendix C):

$$\langle ze_1, zf_1, e_0 f_1, xyz, xye_0, xyf_0\rangle.$$

B.1. Residual Singular Locus and Associated Milnor Number

The would-be singular locus of the proper transform for all n is given by:

$$\{e_0 = 0, x = 0, y = 0\} \cup \{f_0 = 0, x = 0, y = 0\} \cup \{x = 0, y = 0, z = 0\},$$

which is excluded by the SRI. Additionally there is a singularity at:

$$\{x = 0, y = 0, e_1 = 0, a_4 = 0\} \cup \{x = 0, y = 0, f_1 = 0, a_4 = 0\}$$

for $n \geq 4$. Thus PT defines a smooth resolution only for $n = 2, 3$. Note that there are more singular points if one looks only at the hypersurface equation. However they are located in too high codimension compared to our three-dimensional fibration and are not present in our case. The remaining singularities are displayed in table B.2. If one takes a look at the proper transform around these singularities one notices that they take the same form as in the corresponding one brane models. Therefore the singularity parameter a and thereby the Milnor number is the same. As before these residual singularities will be the cause of uncharged localised hypers.

B.2. Analysis of Codimension-One Loci

The model is symmetric in $z_1 \leftrightarrow z_2$. Thus it suffices to only analyse the Σ_1 locus. The resolution divisors $e_0 = 0$ and $e_1 = 0$ are rational fibrations over the locus $z_1 = 0$ in the base, and $\pi^{-1} z_0 = e_0 e_1$. The fiber over $z_1 = 0$ thus consists of two rational curves, given by the vanishing of e_0 and e_1, respectively. Concretely,

$$PT|_{e_0 \to 0} = -e_1 f_1 x^3 + y^2,$$

$$PT|_{e_1 \to 0} = \begin{cases} y^2 - a_4 e_0 f_0 x z^4 - a_6 e_0^2 f_0^2 z^6 & \text{for } a = 2, \\ y^2 - a_4 e_0 f_0 x z^4 & \text{for } a > 2. \end{cases}$$

Thus we have one double intersection at $y^2 = 0$, i.e. Type III which was expected from the Kodaira-Tate table. At this point we recall that we already have analysed a type III fiber (see appendix A.4). Here the situation is very similar and all arguments can be adopted in order to understand how the three gauge bosons of $SU(2)$ come into play in terms of M2-brane wrappings of rational curves in the resolved fiber and dimensional reduction of the M-theory 3-form.

B.3. Analysis of Codimension-Two Loci

Now consider the loci where the residual discriminant hits the gauge brane z_1. This happens at $a_4 = 0$, and the fibers take the form:

$$\begin{aligned}
\mathbb{P}^1_a &: \; PT|_{e_0 \to 0, a_4 \to 0} = -e_1 f_1 x^3 + y^2, \\
\mathbb{P}^1_{b_\pm} &: \; PT|_{e_1 \to 0, a_4 \to 0} = \begin{cases} y^2 - a_6 e_0^2 f_0^2 z^6 & \text{for } a = 2, \\ y^2 & \text{for } a > 2. \end{cases}
\end{aligned}$$

It is allowed to factorize the second equation and take the square root of a_6 because codimension-two loci on the (two-dimensional) base $\mathbb{CP}^2$ are points and therefore $\sqrt{a_6}$ is a single-valued quantity (no branch-cut can be picked up). We are left with three $\mathbb{P}^1$s intersecting once at $y = 0$ ($e_0 = 0 = e_1$). A triple intersection of order one corresponds to type IV in agreement with the expected Kodaira fiber. The resolved fiber is the same as in the one brane case and was displayed in figure A.1. Note that the three $\mathbb{P}^1$s all intersect in one point ($e_0 = e_1 = y = 0$). All other solutions to the above three equations are either forbidden by the SRI or involve vanishing of f_i which does not vanish at this locus.
For $a > 2$ Kodaira's classification applied naïvely to codimension two predicts a type I_0^* fiber. We interpret our findings above as a degenerate, i.e. monodromy reduced, type I_0^* fiber. The double $\mathbb{P}^1_b$ is the node of multiplicity two in the middle, and there is only one further rational curve given by $\mathbb{P}^1_a$, i.e. three nodes of the full I_0^* fiber are deleted.
The weights of the curves under the Cartan of the gauge group are computed analogously to the one brane case. Again we find four fundamental representations at each locus with the usual issue that the fundamental representation of $SU(2)$ is real and therefore we have to count wrapping possibilities which differ only by orientation reversal twice.

Next we look at the intersection of the gauge branes. Naïve application of Kodaira's classification predicts an I_0^* fiber. Let us check this (note that $e_0 \to 0$, $f_1 \to 0$ is forbidden by the SRI):

$$\begin{aligned}
\mathbb{P}^1_a &: \; PT|_{e_0 \to 0,\, f_0 \to 0} = -e_1 f_1 x^3 + y^2, \\
\mathbb{P}^1_{b_{1,2}} &: \; PT|_{e_1 \to 0,\, f_0 \to 0} = y^2, \\
\mathbb{P}^1_c &: \; PT|_{e_1 \to 0,\, f_1 \to 0} = \begin{cases} y^2 - a_4 e_0 f_0 x z^4 - a_6 e_0^2 f_0^2 z^6 & \text{for } a = 2, \\ y^2 - a_4 e_0 f_0 x z^4 & \text{for } a > 2. \end{cases}
\end{aligned}$$

Again we can interpret this as a monodromy reduced I_0^* fiber, with the second line corresponding to the middle $\mathbb{P}^1$ of multiplicity 2, intersecting the two other curves once (see figure B.1).

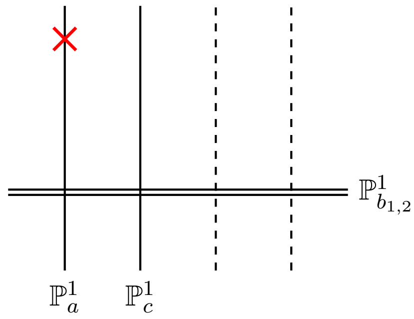


Figure B.1.: Affine Dynkin diagram of the resolved $\{z_1\} \cap \{z_2\}$ locus. The red cross denotes the intersection with the zero-section $z = 0$ of the Weierstrass model.

The next step is to determine the charges of the $\mathbb{P}^1$s under the Cartan $U(1) \times U(1)$. They are given by the intersection product of the $\mathbb{P}^1$s and the resolution divisors E_1 and F_1 defined by $e_1 = 0$ and $f_1 = 0$, respectively.

- Note that the curve $\mathbb{P}^1_a$ is intersected by the zero section. As always, the $\mathbb{P}^1$ intersected by the zero section does not give rise to a new state, but rather to a higher KK state of a ground state accounted for already by the other $\mathbb{P}^1$s in the fiber.

- Consider $\mathbb{P}^1_c$. First note that from the scaling relations in table B.1 we can conclude that $E_1 = -E_0$ and likewise $F_1 = -F_0$, each up to contributions from divisors pulled back from the base which do not affect this computation. This can be used to calculate the following intersection products.
  - $\mathbb{P}^1_c \circ E_1 = [e_1] \cdot [f_1] \cdot [y^2 - e_0 f_0 \ldots] \cdot [e_1] = [e_1] \cdot [f_1] \cdot 2[y] \cdot (-[e_0]) = 0$ since $e_0 f_1$ is in the SRI.
  - $\mathbb{P}^1_c \circ F_1 = [e_1] \cdot [f_1] \cdot [y^2 - e_0 f_0 \ldots] \cdot [f_1] = [e_1] \cdot [f_1] \cdot 2[y] \cdot (-[f_0]) = -2$.

  Therefore the Cartan charges of $\mathbb{P}^1_c$ are $(0, -2)$.

- Consider $\mathbb{P}^1_b$. It is given by the non-reduced object $[e_1]\cdot[f_0]\cdot[y^2]$, so we can think of it as a $\mathbb{P}^1$ with multiplicity 2. The M2-brane can wrap $\mathbb{P}^1_b$ one or two times. Let us decompose $\mathbb{P}^1_b = \mathbb{P}^1_{b_1} + \mathbb{P}^1_{b_2}$. Then $\mathbb{P}^1_{b_1}$ and $\mathbb{P}^1_{b_2}$ are each given by $[e_1] \cdot [f_0] \cdot [y]$. From the scaling relations table B.1 we can conclude that $E_1 = 2Z - F_1 - X$.
  - $\mathbb{P}^1_{b_1} \circ E_1 = [e_1] \cdot [f_0] \cdot [y] \cdot [e_1] = [e_1] \cdot [f_0] \cdot [y] \cdot (2[z] - [f_1] - [x])$. The SRI contains both xyf_0 and ze_1. Therefore $\mathbb{P}^1_{b_1} \circ E_1 = [e_1] \cdot [f_0] \cdot [y] \cdot -[f_1] = -1$.
  - $\mathbb{P}^1_{b_1} \circ F_1 = [e_1] \cdot [f_0] \cdot [y] \cdot [f_1] = 1$.

  Therefore the Cartan charges of $\mathbb{P}^1_{b_{1,2}}$ are $(-1, 1)$.

Now we can wrap $\mathbb{P}^1_{b_i}$ and allowed, i.e. either holomorphic or anti-holomorphic, linear combinations with $\mathbb{P}^1_c$. $\mathbb{P}^1_c$ alone is already wrapped in order to give rise to gauge bosons. The wrappings will combine to a bifundamental representation of $SU(2)\times SU(2)$. The highest weight of the fundamental is $w_0 = 1$ and the weight vector is:

$$\begin{pmatrix} w_0 \\ w_0 + \alpha_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We can easily calculate the Cartan charges of various linear combinations of the $\mathbb{P}^1$s. The $SU(2)_E$ root is $\alpha_1^E = (-2, 0) = \mathbb{P}^1_{b_1} + \mathbb{P}^1_{b_2} + \mathbb{P}^1_c$ where the subscripts E and F distinguish the two $SU(2)$ factors living on Σ_1 and Σ_2. The $SU(2)_F$ root is $\alpha_1^F = (0, -2) = \mathbb{P}^1_c$. The highest weight of the bifundamental representation is $(1, 1)$ which can be represented by the wrapping $-(\mathbb{P}^1_{b_2} + \mathbb{P}^1_c)$. We will see that this choice leads to a consistent picture. Now we can act with α_1^E on it which gives the state $(-1, 1)$ and the curve $\mathbb{P}^1_{b_1}$ is wrapped. In the same way we can act with α_1^F on the first state. Then we obtain the state $(1, -1)$ where the M2-brane is wrapped on $-\mathbb{P}^1_{b_2}$. Finally, acting with both roots on the first state leads to $(-1, -1)$ and wrapping $\mathbb{P}^1_{b_1} + \mathbb{P}^1_c$. We see that all wrappings are either holomorphic or anti-holomorphic linear combinations of the fibral $\mathbb{P}^1$s.

$$\begin{pmatrix} (w_0^E + \alpha_1^E, w_0^F) \\ (w_0^E, w_0^F) \\ (w_0^E, w_0^F + \alpha_1^F) \\ (w_0^E + \alpha_1^E, w_0^F + \alpha_1^F) \end{pmatrix} = \begin{pmatrix} \mathbb{P}^1_{b_1} \\ -(\mathbb{P}^1_{b_2} + \mathbb{P}^1_c) \\ -\mathbb{P}^1_{b_2} \\ \mathbb{P}^1_{b_1} + \mathbb{P}^1_c \end{pmatrix} = \begin{pmatrix} (-1, 1) \\ (1, 1) \\ (1, -1) \\ (-1, -1) \end{pmatrix}.$$

All in all the resolution of the codimension-two singularities of the fibration suggests that there are four states located at $\Sigma_1 \cap \Sigma_2$ which organise into a bifundamental representation of $SU(2) \times SU(2)$. This perfectly matches the prediction from the anomaly constraints.

B.4. Wrap-up

In this chapter we have rewritten our two brane models in Tate form and resolved them. We noticed similarities to the one brane case. Nevertheless it was interesting to see explicitly how the M2-brane wraps the various rational curves at the locus $\Sigma_1 \cap \Sigma_2$ because here both gauge group components are present and thus the M2-branes appearing here are charged under both factors.
The respective `SageMath` code is not written down here because it is very similar to the one brane case (see appendix A.7).

C. Computation of Stanley-Reisner Ideal

One important ingredient of the toric resolution was the computation of the Stanley-Reisner ideal (SRI). It can be computed for every toric space. Its generators describe which variables are not allowed to vanish simultaneously due to the imposed scaling relations. The SRI is not unique. Below I share the `SageMath` code which computes possible SRIs in our cases.

C.1. Type III

```
# Define toric vectors for x y z e0 e1
points = matrix([(1, -1, 0), (0, 1, 0), (-2, -1, 0), (0, 0, 1), (1, 0, 1)])

# Doublecheck that the chosen vectors satisfy the desired scaling relations
print 'Check that chosen vectors are correct:'
2*points[0] + 3*points[1] + points[2]
-points[0] - points[1] - points[3] + points[4]
print

# Compute triangulation
polyhed = Polyhedron(points)
p1 = PointConfiguration(
   points.transpose().augment(vector([0, 0, 0])).transpose())
p1 = p1.restricttostartriangulations((0, 0, 0))
p1 = p1.restricttofinetriangulations()
p1 = p1.restricttoregulartriangulations(True)
tria1 = p1.triangulationslist()
triangl1 = [[i[:-1] for i in j] for j in tria1]

# Display Stanley-Reisner ideals
print 'Stanley-Reisner ideal after blow-up:'
for i in range(len(tria1)):
   faecher = Fan(triangl1[i], points)
   toricx = ToricVariety(faecher, coordinatenames='x y z e0 e1')
   print str(i) + ": " + str(toricx.StanleyReisnerideal())
```

C.2. Type IV

```
# Define toric vectors for x y z e0 e1 e2
points = matrix([(-1, 0, 0), (0, -1, 0), (2, 3, 0),
            (2, 3, 1), (1, 2, 1), (1, 1, 1)])

# Doublecheck that the chosen vectors satisfy the desired scaling relations
```

```
print 'Check that chosen vectors are correct:'
2*points[0] + 3*points[1] + points[2]
- points[0] - points[1] - points[3] + points[4]
- points[0] - 2*points[1] - points[3] + points[5]

polyhed = Polyhedron(points)
p1 = PointConfiguration(
   points.transpose().augment(vector([0, 0, 0])).transpose())
p1 = p1.restricttostartriangulations((0, 0, 0))
p1 = p1.restricttofinetriangulations()
p1 = p1.restricttoregulartriangulations(True)
tria1 = p1.triangulationslist()
triangl1 = [[i[:-1] for i in j] for j in tria1]
print

# Display Stanley-Reisner ideals
print 'Stanley-Reisner ideal after blow-up:'
for i in range(len(tria1)):
   faecher = Fan(triangl1[i], points)
   toricx = ToricVariety(faecher, coordinatenames='x y z e0 e1 e2')
   print str(i) + ": " + str(toricx.StanleyReisnerideal())
```

C.3. Type III × III

```
# Define toric vectors for x y z e0 e1 f0 f1
points = matrix([(1, -1, 0, 0), (0, 1, 0, 0), (-2, -1, 0, 0), (0, 0, 1, 0),
            (1, 0, 1, 0), (0, 0, 0, 1), (1, 0, 0, 1)])
# Doublecheck that the chosen vectors satisfy the desired scaling relations
print 'Check that chosen vectors are correct:'
2*points[0] + 3*points[1] + points[2]
-points[0] - points[1] - points[3] + points[4]
-points[0] - points[1] - points[5] + points[6]
print

# Compute triangulation
polyhed = Polyhedron(points)
p1 = PointConfiguration(
   points.transpose().augment(vector([0, 0, 0, 0])).transpose())
p1 = p1.restricttostartriangulations((0, 0, 0, 0))
p1 = p1.restricttofinetriangulations()
p1 = p1.restricttoregulartriangulations(True)
tria1 = p1.triangulationslist()
triangl1 = [[i[:-1] for i in j] for j in tria1]

# Display Stanley-Reisner ideals
print 'Stanley-Reisner ideal after blow-up:'
for i in range(len(tria1)):
   faecher = Fan(triangl1[i], points)
   toricx = ToricVariety(faecher, coordinatenames='x y z e0 e1 f0 f1')
   print str(i) + ": " + str(toricx.StanleyReisnerideal())
```

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Bibliography

[AGW16] Philipp Arras, Antonella Grassi, and Timo Weigand. "Terminal Singularities, Milnor Numbers and Matter in F-theory". In: *not published yet* (2016).

[Alv83] Luis Alvarez-Gaume. "Supersymmetry and the Atiyah-Singer index theorem". In: *VIIth International Congress on Mathematical Physics (IAMP) Boulder, Colo., August 1-10, 1983.* 1983.

[And+97] L. Andrianopoli et al. "N=2 supergravity and N=2 superYang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map". In: *J. Geom. Phys.* 23 (1997), pp. 111–189. DOI: 10.1016/S0393-0440(97)00002-8. arXiv: hep-th/9605032 [hep-th].

[ASZ84] Orlando Alvarez, I. M. Singer, and Bruno Zumino. "Gravitational Anomalies and the Family's Index Theorem". In: *Commun. Math. Phys.* 96 (1984), p. 409. DOI: 10.1007/BF01214584.

[Avr06] Spyros D. Avramis. "Anomaly-free supergravities in six dimensions". PhD thesis. Natl. Tech. U., Athens, 2006. arXiv: hep-th/0611133 [hep-th].

[AW84] Luis Alvarez-Gaume and Edward Witten. "Gravitational Anomalies". In: *Nucl. Phys.* B234 (1984), p. 269. DOI: 10.1016/0550-3213(84)90066-X.

[BCV14] Andreas P. Braun, Andres Collinucci, and Roberto Valandro. "The fate of U(1)'s at strong coupling in F-theory". In: *JHEP* 07 (2014), p. 028. DOI: 10.1007/JHEP07(2014)028. arXiv: 1402.4054 [hep-th].

[Ber+96] M. Bershadsky et al. "Geometric singularities and enhanced gauge symmetries". In: *Nucl. Phys.* B481 (1996), pp. 215–252. DOI: 10.1016/S0550-3213(96)90131-5. arXiv: hep-th/9605200 [hep-th].

[BG12] Federico Bonetti and Thomas W. Grimm. "Six-dimensional (1,0) effective action of F-theory via M-theory on Calabi-Yau threefolds". In: *JHEP* 05 (2012), p. 019. DOI: 10.1007/JHEP05(2012)019. arXiv: 1112.1082 [hep-th].

[BHV09a] Chris Beasley, Jonathan J. Heckman, and Cumrun Vafa. "GUTs and Exceptional Branes in F-theory - I". In: *JHEP* 01 (2009), p. 058. DOI: 10.1088/1126-6708/2009/01/058. arXiv: 0802.3391 [hep-th].

[BHV09b] Chris Beasley, Jonathan J. Heckman, and Cumrun Vafa. "GUTs and Exceptional Branes in F-theory - II: Experimental Predictions". In: *JHEP* 01 (2009), p. 059. DOI: 10.1088/1126-6708/2009/01/059. arXiv: 0806.0102 [hep-th].

[Bil08] Adel Bilal. “Lectures on Anomalies”. In: (2008). arXiv: 0802.0634 [hep-th].

[BLT12] Ralph Blumenhagen, Dieter Lüst, and Stefan Theisen. *Basic concepts of string theory.* Springer Science & Business Media, 2012.

[Blu+07] Ralph Blumenhagen et al. “Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes”. In: *Phys. Rept.* 445 (2007), pp. 1–193. DOI: 10.1016/j.physrep.2007.04.003. arXiv: hep-th/0610327 [hep-th].

[BS03] Vincent Bouchard and Harald Skarke. “Affine Kac-Moody algebras, CHL strings and the classification of tops”. In: *Adv. Theor. Math. Phys.* 7.2 (2003), pp. 205–232. DOI: 10.4310/ATMP.2003.v7.n2.a1. arXiv: hep-th/0303218 [hep-th].

[Cec10] Sergio Cecotti. “A Geometric Introduction to F-Theory”. 2010.

[CPR97] Philip Candelas, Eugene Perevalov, and Govindan Rajesh. “Toric geometry and enhanced gauge symmetry of F theory / heterotic vacua”. In: *Nucl. Phys.* B507 (1997), pp. 445–474. DOI: 10.1016/S0550-3213(97)00563-4. arXiv: hep-th/9704097 [hep-th].

[Den08] Frederik Denef. “Les Houches Lectures on Constructing String Vacua”. In: *String theory and the real world: From particle physics to astrophysics. Proceedings, Summer School in Theoretical Physics, 87th Session, Les Houches, France, July 2-27, 2007.* 2008, pp. 483–610. arXiv: 0803.1194 [hep-th]. URL: https://inspirehep.net/record/780946/files/arXiv:0803.1194.pdf.

[Erl94] Jens Erler. “Anomaly cancellation in six-dimensions”. In: *J. Math. Phys.* 35 (1994), pp. 1819–1833. DOI: 10.1063/1.530885. arXiv: hep-th/9304104 [hep-th].

[FS90] S. Ferrara and S. Sabharwal. “Quaternionic Manifolds for Type II Superstring Vacua of Calabi-Yau Spaces”. In: *Nucl. Phys.* B332 (1990), pp. 317–332. DOI: 10.1016/0550-3213(90)90097-W.

[FW84] D. Friedan and Paul Windey. “Supersymmetric Derivation of the Atiyah-Singer Index and the Chiral Anomaly”. In: *Nucl. Phys.* B235 (1984), pp. 395–416. DOI: 10.1016/0550-3213(84)90506-6.

[GM00] Antonella Grassi and David R. Morrison. “Group representations and the Euler characteristic of elliptically fibered Calabi-Yau threefolds”. In: (2000). arXiv: math/0005196 [math-ag].

[GM12] Antonella Grassi and David R. Morrison. “Anomalies and the Euler characteristic of elliptic Calabi-Yau threefolds”. In: *Commun. Num. Theor. Phys.* 6 (2012), pp. 51–127. DOI: 10.4310/CNTP.2012.v6.n1.a2. arXiv: 1109.0042 [hep-th].

[GS84] Michael B Green and John H Schwarz. “Anomaly cancellations in supersymmetric D= 10 gauge theory and superstring theory”. In: *Physics Letters B* 149.1-3 (1984), pp. 117–122.

[GSW85] Michael B. Green, John H. Schwarz, and Peter C. West. “Anomaly Free Chiral Theories in Six-Dimensions”. In: *Nucl. Phys.* B254 (1985), pp. 327–348. DOI: 10.1016/0550-3213(85)90222-6.

[Hog12] Robert Hogan. "An Introduction to F-Theory GUT Phenomenology". 2012.

[Hon06] Gabriele Honecker. "Massive U(1)s and heterotic five-branes on K3". In: *Nucl. Phys.* B748 (2006), pp. 126–148. DOI: 10.1016/j.nuclphysb.2006.04.027. arXiv: hep-th/0602101 [hep-th].

[HW97] Amihay Hanany and Edward Witten. "Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics". In: *Nucl. Phys.* B492 (1997), pp. 152–190. DOI: 10.1016/S0550-3213(97)00157-0, 10.1016/S0550-3213(97)80030-2. arXiv: hep-th/9611230 [hep-th].

[IMS97] Kenneth A. Intriligator, David R. Morrison, and Nathan Seiberg. "Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces". In: *Nucl. Phys.* B497 (1997), pp. 56–100. DOI: 10.1016/S0550-3213(97)00279-4. arXiv: hep-th/9702198 [hep-th].

[KMT10a] Vijay Kumar, David R. Morrison, and Washington Taylor. "Global aspects of the space of 6D N = 1 supergravities". In: *JHEP* 11 (2010), p. 118. DOI: 10.1007/JHEP11(2010)118. arXiv: 1008.1062 [hep-th].

[KMT10b] Vijay Kumar, David R. Morrison, and Washington Taylor. "Mapping 6D N = 1 supergravities to F-theory". In: *JHEP* 02 (2010), p. 099. DOI: 10.1007/JHEP02(2010)099. arXiv: 0911.3393 [hep-th].

[Kod63] K. Kodaira. "On Compact Analytic Surfaces: II". In: *Annals of Mathematics* 77.3 (1963), pp. 563–626. ISSN: 0003486X. URL: http://www.jstor.org/stable/1970131.

[Kod68] K. Kodaira. "On the Structure of Compact Complex Analytic Surfaces, III". In: *American Journal of Mathematics* 90.1 (1968), pp. 55–83. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/2373426.

[KT09] Vijay Kumar and Washington Taylor. "A Bound on 6D N=1 supergravities". In: *JHEP* 12 (2009), p. 050. DOI: 10.1088/1126-6708/2009/12/050. arXiv: 0910.1586 [hep-th].

[KT11] Vijay Kumar and Washington Taylor. "String Universality in Six Dimensions". In: *Adv. Theor. Math. Phys.* 15.2 (2011), pp. 325–353. DOI: 10.4310/ATMP.2011.v15.n2.a3. arXiv: 0906.0987 [hep-th].

[KV97] Sheldon H. Katz and Cumrun Vafa. "Matter from geometry". In: *Nucl. Phys.* B497 (1997), pp. 146–154. DOI: 10.1016/S0550-3213(97)00280-0. arXiv: hep-th/9606086 [hep-th].

[MV96a] David R. Morrison and Cumrun Vafa. "Compactifications of F theory on Calabi-Yau threefolds. 1". In: *Nucl. Phys.* B473 (1996), pp. 74–92. DOI: 10.1016/0550-3213(96)00242-8. arXiv: hep-th/9602114 [hep-th].

[MV96b] David R. Morrison and Cumrun Vafa. "Compactifications of F theory on Calabi-Yau threefolds. 2." In: *Nucl. Phys.* B476 (1996), pp. 437–469. DOI: 10.1016/0550-3213(96)00369-0. arXiv: hep-th/9603161 [hep-th].

[MW15] Luca Martucci and Timo Weigand. "Non-perturbative selection rules in F-theory". In: *JHEP* 09 (2015), p. 198. DOI: 10.1007/JHEP09(2015)198. arXiv: 1506.06764 [hep-th].

[Nak03] Mikio Nakahara. *Geometry, topology and physics.* CRC Press, 2003.

[NS97] Hitoshi Nishino and Ergin Sezgin. "New couplings of six-dimensional supergravity". In: *Nucl. Phys.* B505 (1997), pp. 497–516. DOI: 10.1016/S0550-3213(97)00357-X. arXiv: hep-th/9703075 [hep-th].

[RT08] M. Rossi and L. Terracini. "MAPLE subroutines for computing Milnor and Tyurina numbers of hypersurface singularities with application to Arnol'd adjacencies". In: *ArXiv e-prints* (Sept. 2008). arXiv: 0809.4345 [math.AG].

[Sag92] Augusto Sagnotti. "A Note on the Green-Schwarz mechanism in open string theories". In: *Phys. Lett.* B294 (1992), pp. 196–203. DOI: 10.1016/0370-2693(92)90682-T. arXiv: hep-th/9210127 [hep-th].

[Sla81] R. Slansky. "Group Theory for Unified Model Building". In: *Phys. Rept.* 79 (1981), pp. 1–128. DOI: 10.1016/0370-1573(81)90092-2.

[SS04] Claudio A. Scrucca and Marco Serone. "Anomalies in field theories with extra dimensions". In: *Int. J. Mod. Phys.* A19 (2004), pp. 2579–2642. DOI: 10.1142/S0217751X04018518. arXiv: hep-th/0403163 [hep-th].

[SW16] Sakura Schäfer-Nameki and Timo Weigand. "F-theory and 2d $(0,2)$ theories". In: *JHEP* 05 (2016), p. 059. DOI: 10.1007/JHEP05(2016)059. arXiv: 1601.02015 [hep-th].

[Tat75] John Tate. "Algorithm for determining the type of a singular fiber in an elliptic pencil". In: *Modular functions of one variable IV.* Springer, 1975, pp. 33–52.

[Tay11] Washington Taylor. "TASI Lectures on Supergravity and String Vacua in Various Dimensions". In: (2011). arXiv: 1104.2051 [hep-th].

[Vaf05] Cumrun Vafa. "The String landscape and the swampland". In: (2005). arXiv: hep-th/0509212 [hep-th].

[Vaf96] Cumrun Vafa. "Evidence for F theory". In: *Nucl. Phys.* B469 (1996), pp. 403–418. DOI: 10.1016/0550-3213(96)00172-1. arXiv: hep-th/9602022 [hep-th].

[Wei10] Timo Weigand. "Lectures on F-theory compactifications and model building". In: *Class. Quant. Grav.* 27 (2010), p. 214004. DOI: 10.1088/0264-9381/27/21/214004. arXiv: 1009.3497 [hep-th].

[Wit96] Edward Witten. "Nonperturbative superpotentials in string theory". In: *Nucl. Phys.* B474 (1996), pp. 343–360. DOI: 10.1016/0550-3213(96)00283-0. arXiv: hep-th/9604030 [hep-th].

[WZ71] Julius Wess and Bruno Zumino. "Consequences of anomalous Ward identities". In: *Physics Letters B* 37.1 (1971), pp. 95–97.

[Zum83] Bruno Zumino. "Chiral anomalies and differential geometry: lectures given at Les Houches, August 1983". In: *Relativity, groups and topology: Proceedings, 40th Summer School of Theoretical Physics - Session 40.* [,361(1983)]. 1983, pp. 1291–1322. URL: http://www.osti.gov/energycitations/product.biblio.jsp?osti_id=5461614.

[ZWZ84] Bruno Zumino, Yong-Shi Wu, and A. Zee. "Chiral Anomalies, Higher Dimensions, and Differential Geometry". In: *Nucl. Phys.* B239 (1984), pp. 477–507. DOI: 10.1016/0550-3213(84)90259-1.

Erklärung

Ich versichere, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 27.10.2016,
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