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Using Computational Algebraic Geometry in F-Theory

Bachelor Thesis in Physics
submitted by

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Abstract

Opposed to type IIB string theory F-theory provides a method to describe strings in a non-pertubative way. This approach is purely geometric and therefore predestinated to be examined by the tools of algebraic geometry.
In this work an introduction to algebraic geometry from a computational point of view is provided. After defining varieties and collecting their most important properties, Groebner bases are defined which are fundamental for all computations with ideals.
Finally, a specific F-theory model is analysed with the help of `Singular`.

Zusammenfassung

Im Gegensatz zu Typ-IIB-Stringtheorie werden Strings in der F-Theorie nicht-pertubativ beschrieben. Dieser Zugang ist rein geometrisch und damit prädestiniert um mit den Werkzeugen der algebraischen Geometrie behandelt zu werden.
In dieser Arbeit werden die Grundlagen der algebraischen Geometrie eingeführt. Nachdem Varietäten definiert und ihre wichtigsten Eigenschaften zusammengetragen werden, wird ein kurzer Blick auf Gröbnerbasen geworfen. Auf diesem Konzept basieren die Algorithmen der Computeralgebrasysteme.
Schließlich wird mit Hilfe von `Singular` ein gegebenes F-Theorie-Modell analysiert.

This Bachelor Thesis has been carried out by Philipp Arras at the
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1 Physical Context: F-Theory

In this section I will briefly explain the physical context in which the issue of decomposing algebraic varieties has occurred.
Since the standard formulation of type IIB string theory is pertubative it is very difficult to arrive at non-pertubative statements. For this purpose one uses a different approach, namely F-theory. Here all physical content is described by geometric means.
For general background on string theory I refer to [Gre12a], [Gre12b] and [BLT13]. An introduction to F-theory is given e.g. in [Den08] and [MP13].
In our case we start from type IIB orientifold string theory on M^6/σ where M^6 is a 6-dimensional Calabi-Yau manifold and $\sigma : M^6 \to M^6$ denotes a holomorphic involution. Then M^6/σ is not a Calabi-Yau manifold any more since the involution has fix points in which the space is not smooth and the Ricci-curvature not trivial.
All in all the idea is that the whole spacetime is $\mathbb{R}^{1,3}$ times a non-trivial 6-dimensional internal space M^6/σ (Figure 1.1).

Figure 1.1: Compactified space

Now, amongst others gravity (i.e. a massless spin-2-state) can be recovered in the closed strings excitations at a massless level whereas matter and Yang-Mills gauge bosons appear in the spectrum of open strings. Both ends of such an open string are attached to so-called Dp-branes. A Dp-brane is a (p+1)-dimensional subspace of $\mathbb{R}^{1,3} \times M^6/\sigma$.
In type IIB string theory only Dp-branes with odd p appear. In the sequel, we focus on spacetime-filling D7-branes. They fill $\mathbb{R}^{1,3}$ completely and a 4-dimensional subspace of M^6 called a 4-cycle. There are configuration where N D7-branes are located on top of each other (see Figure 1.2). There are N^2 possibilities to attach an open string to this set-up. Quantizing the strings leads to N^2 vector states A^μ in the adjoint representation of $SU(N)$. This is how we can obtain $SU(N)$ gauge theory in Type IIB string theory. Additionally, the D_n series can be produced as gauge group.
Unfortunately, a generalization to all other classical Lie algebras is not immediate. At this point, F-theory comes into play. It can reproduce all classical Lie groups as gauge groups. The gauge groups which could not be used in type IIB string theory (e.g. E_6, E_7 or E_8)

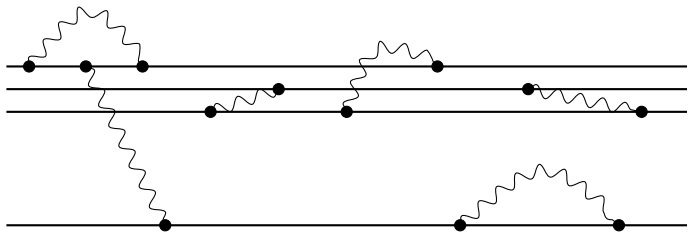


Figure 1.2: $SU(N)$ gauge theory

require bound states of strings, a non-pertubative effect.
The set-up in F-theory is a bit different as opposed to type IIB string theory. The space M^6/σ is replaced by the smooth space B_6 and an elliptic fibration over it. B_6 is not Calabi-Yau as well. Descriptively, the non-smooth fixpoints (in M^6/σ) transition into smooth points at the cost of introducing curvature. So B_6 is not Calabi-Yau as well. However, the fibration is such that its curvature cancels exactly the curvature of B_6. Therefore the total $(6+2)$-real-dimensional space is a Calabi 4-fold. The physics is extracted by analysing the fibration. Where it degenerates the 7-branes are localised. This will be explicitly seen in an example (Chapter 3).
Let us exemplify what can happen. Consider two stacks of 7-branes which intersect transversally. Let the first stack consist of two and the second stack of three 7-branes. Now, open strings can be attached to those five branes. There are three possibilities. First, the open string is attached at both ends to the first stack. This gives rise to $2^2 = 4$ $U(2)$ gauge bosons. Second, the string is attached to the second stack which corresponds to $3^2 = 9$ $U(3)$ gauge bosons. And third, one end is attached to a brane of the first stack and the other end to one of the second. This is depicted in Figure 1.3.

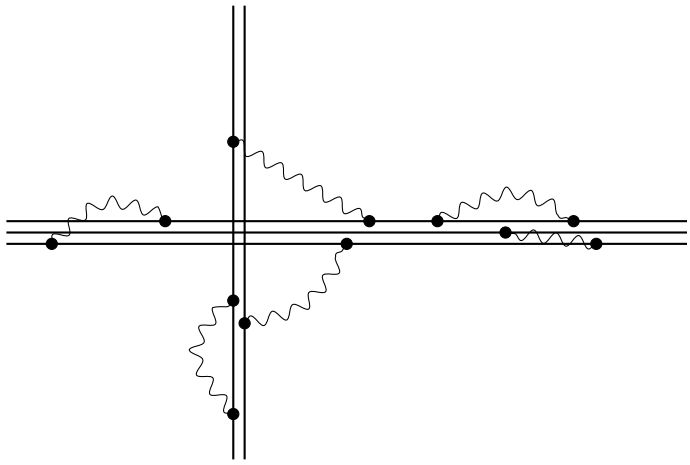


Figure 1.3: Intersection of 7-branes and open strings attached to them

Due to the tension of a string and the fact that string ends can move on branes the latter string will contract and be localised at the intersection of the two brane stacks. This leads to massless chiral or anti-chiral multiplets which are charged under $U(2)$ and $U(3)$. More

precisely, the fermions live in the bifundamental representation of $U(2) \times U(3)$. This is denoted by $(2, 3)$.
Therefore the gauge bosons live on the entire brane and the charged matter on a sublocus of the intersection of the respective branes.

Gauge bosons	$\leftrightarrow$	entire brane
charged matter	$\leftrightarrow$	sublocus of intersection of branes

Note. It can happen that one single 7-brane intersects transversally itself. This is also physically important and occurs in the latter computation.

Side remark. In the sequel, we will work with polynomials to examine the geometric properties of the given situation in B_6. Therefore it is useful to talk about complex dimensions and not about real dimensions since the dimension associated to a polynomial will correspond to the complex dimension. This will be explained in Chapter 2.
That the above procedure is possible is not in the least clear. A rough explanation is: In the simplest case the internal space is a Calabi-Yau manifold which is by definition endowed with a complex structure. In our case, we are working with spacetime-filling 7-branes. Branes have to respect supersymmetry and therefore the complex structure on B_6.
Therefore it is possible to work with complex polynomials which is of great help as we are now working over an algebraically closed field ($\mathbb{C}$). Algebraic geometry over not algebraically closed fields is generally very tricky.

Interactions. We have seen that the transversal intersection of two 7-branes gives rise to matter. This transversal intersection has codimension 2 and is therefore a complex curve.[1] It can happen that two or three of these curves again intersect. The intersection locus has generically codimension 3, i.e. is a point in the compact space. Some properties of this 0-dimensional intersection locus will determine the properties of this interaction.
More on the physical context of F-theory can be found in [Wei10].

Aim. The aim of this work is to compute the intersection loci of a specific F-theory set-up. How this comes about can be found in [Bor+14]. In a nutshell, the matter curves on the compact manifold are given in form of polynomial equations in multiple variables. Based on this the loci where three matter curves intersect each other shall be computed. Finally, the enhancement of the elliptic fibration in those loci shall be analysed.

[1] A 7-brane has 8 real dimensions. 4 of them are in $\mathbb{R}^{1,3}$ and 4 in the compact space. The compact space has 3 complex dimensions and the part of the 7-brane has 2 complex dimensions. Therefore the 7-brane has codimension 1 and a transversal intersection is generically one codimension lower.

2 Mathematical Prerequisites

The aim of this chapter is to gain some intuition about varieties and to understand how to use a computer algebra system like `Singular`. Therefore not all statements will be proven. When omitting important proofs I will give a reference to where the proof can be found.
We start from the very basic geometric and algebraic definitions. After describing the Algebra-Geometry Dictionary which allows to literally translate algebraic statements into geometric ones and vice versa, I will elaborate on the notion of dimension. This concept is a bit more complicated than in the case of manifolds. The set of points where something different happens will be called singular locus.
In the last section a short summary of the computational methods in this context is given.
Generally, algebraic geometry can be done over different fields. They can be classified into finite or infinite fields and algebraically closed or not algebraically closed fields. In the previous chapter I have described that our given environment are zero sets of complex polynomials. Therefore I will concentrate on infinite algebraically closed fields and indicate at which points problems can arise when considering more general fields.
Finally, a short introduction on computational aspects is given. It is beyond the scope of this work to describe the algorithms. Instead, the basic object, namely the Gröbner basis, is introduced and its most important property stated.
The reference of choice for the link between algebraic geometry and computations is [CLO10]. Parts of formulations of definitions, theorems and ideas for some proofs are taken from this reference.

2.1 Ideals and Varieties

Algebraic geometry heavily relies on the fact that there is a strong connection between ideals being algebraic objects and varieties being geometric ones. We will start with the basic definitions and arrive via *Hilbert's Nullstellensatz* at the Algebra-Geometry dictionary. Already at this point, I would like to stress that it holds only for ideals in polynomial rings over algebraically closed fields. Without this assumption algebraic geometry is far from being intuitive and descriptive.
After defining an adequate topology we decompose varieties uniquely into irreducible ones. Unfortunately, this does not translate into the language of ideals in full generality: Only radical ideals can be decomposed into prime ideals uniquely. For non-radical ideals we need a more sophisticated concept: the primary decomposition.

2.1.1 Basic Definitions [CLO10]

Let us start from the definition of polynomial rings.

Definition 1. *Let R be a ring. Then $R[X]$ is the set of all finite sequences*

$$R^{(\mathbb{N}_0)} := \{(a_i)_{i\in\mathbb{N}_0} \ : \ a_i \in R, a_i = 0 \text{ for almost all } i\}.$$

We define an addition component-by-component ($(a_i)_{i\in\mathbb{N}_0} + (b_i)_{i\in\mathbb{N}_0} := (a_i + b_i)_{i\in\mathbb{N}_0}$) and a multiplication by the convolution of both sequences ($(a_i)_{i\in\mathbb{N}_0} \cdot (b_i)_{i\in\mathbb{N}_0} := (\sum_{i_0}^{k})a_i b_{k-i}$).
The resulting ring is denoted by $R[X]$ and called polynomial ring over R. *Let us define $x := (0, 1, 0, 0, \ldots)$. Then*

$$(\underbrace{0, \ldots, 0}_{k \text{ zeros}}, 1, 0, \ldots) \equiv x^k = \underbrace{x \cdot \ldots \cdot x}_{k \text{ times}}$$

and all Elements $f \in R^{(\mathbb{N}_0)}$ can be written as

$$f = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

Definition 2. *Let R be a ring. The* polynomial ring in n variables over R *is defined recursively:*

$$R[x_1, \ldots, x_n] := R[x_1, \ldots, x_{n-1}][x_n].$$

The elements of $R[x_1, \ldots, x_n]$ can be written as

$$\sum_{k=(k_1,\ldots,k_n)\in\mathbb{N}^n} a_k \, x_1^{k_1} \cdots x_n^{k_n}.$$

In the sequel, we will only consider polynomial rings over fields, e.g. $\mathbb{R}$ or $\mathbb{C}$. To avoid endless repetitions let k be a field from now on. We move on to the basic algebraic object of this work.

Definition 3. *A subset $I \subset k[x_1, \ldots, x_n]$ is an* ideal *if:*

1. $0 \in I$.
2. $f, g \in I \ \Rightarrow \ f + g \in I$.
3. $f \in I$ *and* $h \in k[x_1, \ldots, x_n] \ \Rightarrow \ hf \in I$.

Lemma 4. *If $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$, then*

$$\langle f_1, \ldots, f_s \rangle := \left\{ \sum_{i=1}^{s} h_i f_i \ : \ h_1, \ldots, h_s \in k[x_1, \ldots, x_n] \right\}$$

is an ideal of $k[x_1, \ldots, x_n]$ and is called the ideal generated by the f_is.

Proof. [CLO10, p. 30] □

This fact leads to the first interesting observation. When manipulating systems of linear equations one is allowed to multiply any equation by a polynomial and to add two equations. These rules correspond to the definition of an ideal. Therefore we can think of the ideal $\langle f_1, \ldots, f_s \rangle$ as the set of all consequences of the equations $f_1 = f_2 = \ldots = f_s = 0$ in the sense that if all generating polynomials are set to zero then all elements of the ideal will be zero as well.
We now come to special ideals which will play an important role in the context of varieties.

Definition 5. *An ideal I is* radical *if the following implication holds:*

$$f^m \in I \text{ for some integer } m \geq 1 \quad \Rightarrow \quad f \in I.$$

Given an arbitrary ideal it is always possible to produce a radical ideal.

Definition 6. *Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. The* radical *of I is*

$$\sqrt{I} := \{f \ : \ f^m \in I \text{ for some integer } m \geq 1\}.$$

Lemma 7. *$\sqrt{I}$ is a radical ideal and $I \subset \sqrt{I}$.*

Proof. [CLO10, p. 176] □

To see that equality is not guaranteed consider $I = \langle x^2 \rangle \subset \mathbb{C}[x]$. Then $\sqrt{I} = \langle x \rangle \neq \langle x^2 \rangle$.
We move on to another important property an ideal can have.

Definition 8. *An ideal $I \subset k[x_1, \ldots, x_n]$ is* prime *if the following implication is true:*

$$f, g \in k[x_1, \ldots, x_n] \text{ and } fg \in I \quad \Rightarrow \quad f \in I \text{ or } g \in I.$$

Ideals can be combined in the following three ways: the sum, the product and the intersection.

Definition 9. *Let I and J be ideals in $k[x_1, \ldots, x_n]$. Then the* sum *is defined as*

$$I + J := \{f + g \ : \ f \in I \text{ and } g \in J\}.$$

The product *IJ is defined to be the ideal generated by all polynomials $f \cdot g$ where $f \in I$ and $g \in J$.*
The intersection *$I \cap J$ of two ideals I and J in $k[x_1, \ldots, x_n]$ is the set of all polynomials belonging to both I and J.*

Note that the set $\{f \cdot g \ : \ f \in I, g \in J\}$ does not need to be an ideal and that we have to show that $I + J$ and $I \cap J$ are ideals.

Example 10. *Consider* $I = \langle x, y\rangle \subset k[x, y]$. *Then the set* $\{\langle x, y\rangle\}^2 = \{f \cdot g \ : \ f, g \in \langle x, y\rangle\}$ *is not an ideal.*[1]

Proposition 11. *Let* $I = \langle f_1, \ldots, f_r\rangle$ *and* $J = \langle g_1, \ldots, g_s\rangle$ *be ideals in* $k[x_1, \ldots, x_n]$. *Then* $I + J$ *is the smallest ideal containing both* I *and* J *and one has*

$$I + J = \langle f_1, \ldots, f_r, g_1, \ldots, g_s\rangle.$$

For the product it suffices to take only the product of the generating polynomials:

$$IJ = \langle f_i g_j \ : \ 1 \leq i \leq r, 1 \leq j \leq s\rangle.$$

Proof. [CLO10, pp. 183, 185] □

Corollary 12. *If* $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$, *then*

$$\langle f_1, \ldots, f_r\rangle = \langle f_1\rangle + \ldots + \langle f_r\rangle.$$

Proof. See [CLO10, p. 184] □

Proposition 13. $I \cap J$ *is an ideal for* I *and* J *being ideals.*

Proof. [CLO10, p. 186] □

Proposition 14. *Let* I, J *be ideals. Then* $IJ \subset I \cap J$.

Proof. Let $I = \langle f_1, \ldots, f_s\rangle$ and $J = \langle g_1, \ldots, g_l\rangle$. Let $f \in IJ$. Then by Proposition 11 there exist $c_{ij} \in k[x_1, \ldots, x_n]$ such that

$$f = \sum_{i,j} \underbrace{c_{ij} f_i}_{=:\tilde{c}_{ij}} g_j = \sum_{i,j} \tilde{c}_{ij} g_j.$$

Therefore, $f \in J$ and likewise for I. □

Note that we have only proven the proposition for finitely generated ideals. However, this is not a restriction because every ideal in a polynomial ring is finitely generated (as we will see in Theorem 78).
Now, we define the basic geometric object of all analysis in this work.

[1] x^2, y^2 are in $\{\langle x, y\rangle\}^2$ but $x^2 + y^2 \notin \{\langle x, y\rangle\}^2$. This can be seen as follows. Suppose $x^2 + y^2 \in \{\langle x, y\rangle\}^2$. Then there exist $A_1, \ldots, A_4 \in k[x, y]$ with $x^2 + y^2 = (A_1 x + A_2 y)(A_3 x + A_4 y)$. But then

$$x^2 + y^2 = A_1 A_3 x^2 + (A_1 A_4 + A_2 A_3)xy + A_2 A_4 y^2.$$

Thus, $A_1 A_4 + A_2 A_3 = 0$ and $x^2 + y^2 = A_1 A_3 (x^2 - y^2)$. Therefore $(1 - A_1 A_3)x^2 + (1 + A_1 A_3)y^2 = 0$ which implies both $A_1 A_3 = 1$ and $A_1 A_3 = -1$ which is clearly a contradiction.

Definition 15. *Let k be a field and let $f_1, \dots, f_s$ be polynomials in $k[x_1, \dots, x_n]$. Then we set*

$$\mathbf{V}(f_1, \dots, f_s) = \{p \in k^n \ : \ f_i(p) = 0 \text{ for all } 1 \leq i \leq s\}.$$

We call $\mathbf{V}(f_1, \dots, f_s)$ the affine variety *defined by $f_1, \dots, f_s$.*

Note. The above definition is the definition of *affine varieties* in contrast to *projective varieties*. In this abstract we will only work with affine varieties and will drop the "affine" from now on.

On the difference between manifolds and varieties. The relationship of varieties and manifolds can be recognized directly in the definition. Manifolds can be described by equations in the above way but via the Implicit Function Theorem one additionally needs that the rank of the Jacobian matrix is constant along the zero set. Here we do not require this.
For varieties so-called singular points are allowed which lead to a non-constant dimension. Besides, varieties possibly consist of several components with different dimension. Both does not occur when considering manifolds.
It seems that varieties have less structure than manifolds. Actually, the opposite is the case. However, I leave it as a side remark since it is not essential in the sequel.[a]
Finally, be aware that amongst others in French the expression *variété* is used for both varieties and manifolds.

[a] The difference comes along with two different notions of isomorphism. Two manifolds are called isomorphic if there exist a diffeomorphism between them. Two varieties are called isomorphic if there exists a polynomial maps giving rise to mutually inverse bijections.
It turns out that the only diffeomorphism invariant of a compact connected (2-real-dimensional) surface is the genus. However, smooth connected projective (1-complex-dimensional) curves over $\mathbb{C}$ (which are compact connected surfaces if we forget about their variety structure) are not determined by the genus g.

- For $g \geq 2$: there exists a $6(g-1)$-dimensional family of non-isomorphic curves of genus g.
- There exists a 2-dimensional family of non-isomorphic curves of genus 1.
- For $g = 0$ the variety structure is uniquely determined.

Just as with ideals two varieties can be combined.

Lemma 16. *If $V, W \subset k^n$ are affine varieties, then so are $V \cup W$ and $V \cap W$.*

Proof. Suppose $V = \mathbf{V}(f_1, \dots, f_s)$ and $W = \mathbf{V}(g_1, \dots, g_t)$. It is clear that $V \cap W = \mathbf{V}(f_1, \dots, f_s, g_1, \dots, g_t)$.

Claim: $V \cup W = \mathbf{V}(f_i g_j \ : \ 1 \leq i \leq s, 1 \leq j \leq t)$

"$\subset$": Let $p \in V$. That means $f_i(p) = 0$ for all i. Therefore $f_i(p) \cdot g_j(p) = 0$ for all i and j. This means $V \subset \mathbf{V}(f_i g_j)$. Similarly, one can show that $W \subset \mathbf{V}(f_i g_j)$.
"$\supset$": Let $a \in \mathbf{V}(f_i g_j)$. Then two cases can occur:

1. $p \in V$. Then we are done.

2. $p \notin V$. Then there exist an i_0 such that $f_{i_0}(p) \neq 0$. But we have required that $f_i(p)g_j(p) = 0$ for all i including i_0. Therefore $g_j(p)$ has to vanish for all j, i.e. $p \in W$.

Altogether, we have shown that p is either in V or in W. #

This shows the lemma. □

Now, we get a small taste of the ideal-variety correspondence. If two sets of polynomials generate the same ideal, we will get the same vanishing set for both sets.

Proposition 17. *If* $\langle f_1, \ldots, f_s \rangle = \langle g_1, \ldots, g_l \rangle$ *then* $\mathbf{V}(f_1, \ldots, f_s) = \mathbf{V}(g_1, \ldots, g_l)$.

Proof. Let $p \in \mathbf{V}(f_1, \ldots, f_s)$. Thus, $f_i(p) = 0$ for $1 \leq i \leq s$. By assumption, every element in $\langle g_1, \ldots, g_l \rangle$ can be written as linear combination of $f_1, \ldots, f_s$. Use this fact for the basis elements: There exist polynomials $c_{ij} \in k[x_1, \ldots, x_n]$ such that $g_k = \sum_{j=1}^{s} c_{kj} f_j$ for all $1 \leq k \leq l$. Plugging in p yields

$$g_k(p) = \sum_j c_{ij}(p) \underbrace{f_j(p)}_{=0} = 0 \text{ for all } 1 \leq k \leq l.$$

Therefore $p \in \mathbf{V}(g_1, \ldots, g_l)$. □

It seems that Proposition 17 is very intuitive. But note that the converse implication of Proposition 17 is not true. It is not possible to read off of a variety uniquely its defining polynomials. A counterexample is given later (Example 41).
The crucial conclusion is that our first impression has to be revised. We thought of varieties as being defined by a set of polynomial equations. But now we see that a variety cannot be characterized uniquely by a set of equations. There may be different sets of equations which describe the same variety. So we have to ask what the underlying object really is. The answer is connected to the map $\mathbf{V} : \{\text{Ideals}\} \to \{\text{Varieties}\}$. The questions is whether or not an inverse map exists. This would then be called *ideal-variety correspondence.*

Definition 18. *Let* $V \subset k^n$ *be a variety. Then we set*

$$\mathbf{I}(V) = \{f \in k[x_1, \ldots, x_n] \ : \ f(a_1, \ldots, a_n) = 0 \textit{ for all } (a_1, \ldots, a_n) \in V\}$$

and call $\mathbf{I}(V)$ the ideal of V.

Lemma 19. $\mathbf{I}(V) \subset k[x_1, \ldots, x_n]$ *is an ideal.*

Proof. It is obvious that $0 \in \mathbf{I}(V)$ since zero polynomials vanish always by definition. For $f, g \in \mathbf{I}(V)$, $h \in k[x_1, \ldots, x_n]$ and $(a_1, \ldots, a_n) \in V$ arbitrary we have

$$\begin{aligned} f(a_1, \ldots, a_n) + g(a_1, \ldots, a_n) &= 0 + 0 = 0 \\ \text{and } h(a_1, \ldots, a_n) \cdot f(a_1, \ldots, a_n) &= h(a_1, \ldots, a_n) \cdot 0 = 0. \end{aligned}$$

□

To gain some intuition let us look at some examples.

Example 20.

- $\mathbf{I}(\{(0,0)\}) = \langle x, y\rangle$
- *Let k be infinite. Then* $\mathbf{I}(k^n) = \{0\}$
- *Let* $I = \langle y - x^2, z - x^3\rangle$. *Then* $\mathbf{I}(\mathbf{V}(I)) = I$.
- *But:* $J = \langle (y - x^2)^2, z - x^3\rangle$. *Then still* $\mathbf{I}(\mathbf{V}(J)) = I$.

The proof of the last two examples is quite tedious and since we will arrive at a general statement I shall not give the proof here.
The important fact is that the maps $\mathbf{I}$ and $\mathbf{V}$ cannot be inverse maps since there exist ideals $I \neq J$ with $\mathbf{I}(\mathbf{V}(J)) = I$ as stated in Example 20. The intuitive way is to restrict the maps to some subset of all ideals or to some subset of all varieties. However, the following lemma shows that it would make no difference to exclude some varieties.

Lemma 21. *If* $f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ *then* $\langle f_1, \ldots, f_s\rangle \subset \mathbf{I}(\mathbf{V}(f_1, \ldots, f_s))$.

Proof. Let $f \in \langle f_1, \ldots, f_s\rangle$. In other words: $f = \sum_{i=1}^{s} h_i f_i$ for some $h_i \in k[x_1, \ldots, x_n]$. Since all $f_1, \ldots, f_s$ vanish on $\mathbf{V}(f_1, \ldots, f_s)$, f vanishes as well on this set. Therefore $f \in \mathbf{I}(\mathbf{V}(f_1, \ldots, f_s))$. □

Therefore, we have to restrict to some subset of all ideals. The precise statement is developed in the next paragraph. But first, we collect another important result.

Proposition 22. *Let V, W be varieties in k^n. Then:*

1. $V \subset W \Leftrightarrow \mathbf{I}(V) \supset \mathbf{I}(W)$ *and*
2. $V = W \Leftrightarrow \mathbf{I}(V) = \mathbf{I}(W)$.

Proof. [CLO10, p. 35]. □

Note that the above problem of squaring a generator does not arise here.

2.1.2 Hilbert's Nullstellensatz and Ideal-Variety Correspondence [CLO10]

Beginning with Hilbert's Nullstellensatz we now carefully establish the connection between ideals and varieties.

Lemma 23 (Weak Nullstellensatz)**.** *Let k be an algebraically closed field and $I \subset k[x_1, \ldots, x_n]$ be an ideal with $\mathbf{V}(I) = \varnothing$. Then $I = k[x_1, \ldots, x_n]$.*

Proof. [CLO10, pp. 170–172]. □

The proof of the *Weak Nullstellensatz* is relatively complicated and technical. The important thing is that it needs the restriction to algebraically closed fields.

Theorem 24 (Hilbert's Nullstellensatz)**.** *Let be $f, f_1, \ldots, f_s \in k[x_1, \ldots, x_n]$ such that $f \in \mathbf{I}(\mathbf{V}(f_1, \ldots, f_s))$. Then there exists an integer $m \geq 1$ such that $f^m \in \langle f_1, \ldots, f_s \rangle$ (and conversely).*

Proof. Let f be a non-zero polynomial which vanishes at every common zero of the polynomials $f_1, \ldots, f_s$. To show the theorem consider the ideal $\hat{I} \subset k[x_1, \ldots, x_n, y]$ defined by $\hat{I} = \langle f_1, \ldots, f_s, 1 - yf \rangle$.

Claim: $\mathbf{V}(\hat{I}) = \varnothing$

Let $(a_1, \ldots, a_{n+1}) \equiv (a, a_{n+1}) \in k^{n+1}$. Then there are two cases.

1. $a \in \mathbf{V}(f_1, \ldots, f_s)$, i.e. a is a common zero of $f_1, \ldots, f_s$. So $f(a) = 0$ and

$$(1 - yf)(a_1, \ldots, a_{n+1}) = 1 - a_{n+1} \cdot \underbrace{f(a)}_{=0} = 1 \neq 0.$$

   Therefore $(a_1, \ldots, a_{n+1}) \notin \mathbf{V}(\hat{I})$.

2. $a \notin \mathbf{V}(f_1, \ldots, f_s)$, i.e. a is not a common zero of $f_1, \ldots, f_s$. So there exists a j with $f_j(a) \neq 0$. Now, think of f_j as a polynomial in $n+1$ variables which simply does not depend on the last one. Then $f_j(a, a_{n+1}) = f_j(a) \neq 0$ and $(a, a_{n+1}) \notin \mathbf{V}(\hat{I})$.

#

By the *Weak Nullstellensatz* (Lemma 23) one has $\hat{I} = k[x_1, \ldots, x_n, y]$ including $1 \in \hat{I}$. This means that 1 can be written as linear combination of the basis polynomials, i.e. there exist polynomials $p_i, q \in k[x_1, \ldots, x_n, y]$ with

$$1 = \sum_{i=1}^{s} p_i(x_1, \ldots, x_n, y) \cdot f_i + q(x_1, \ldots, x_n, y) \cdot (1 - yf).$$

Set $y = 1/f$ which is well-defined since $f = f(x_1, \ldots, x_n)$ is constant in y. This yields

$$1 = \sum_{i=1}^{s} p_i(x_1, \ldots, x_n, 1/f) f_i.$$

Multiplying by f^m with $m \in \mathbb{N}$ large enough yields $f^m = \sum_{i=1}^{s} A_i f_i$ with $A_i \in k[x_1, \ldots, x_n]$. So $f^m \in \langle f_1, \ldots, f_s \rangle$ for some $m \in \mathbb{N}$. □

The above theorem is the reason why we have defined radical ideals.
Our wish is to have a one-to-one correspondence between varieties and ideals. Our map $\mathbf{I}$ produces only radical ideals. With our previous knowledge (Lemma 21) we need an ideal that is contained in a given ideal I which is invariant under $\mathbf{I}(\mathbf{V}(\cdot))$.
We remember that the radical of an ideal is a radical ideal (Lemma 7) and add the following proposition to gain good intuition about what the radical of an ideal is.

Proposition 25. *Let $f \in k[x_1, \ldots, x_n]$ and $f = cf_1^{a_1} \cdots f_r^{a_r}$ the factorisation of f into a product of distinct irreducible polynomials. Then*

$$\sqrt{\langle f \rangle} = \langle f_1 \cdots f_r \rangle.$$

Proof. [CLO10, p. 180] □

We can restate the Nullstellensatz.

Theorem 26 (The Strong Nullstellensatz)**.** *Let k be an algebraically closed field and $I \subset k[x_1, \ldots, x_n]$ an ideal. Then*

$$\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}.$$

Proof. Two implications are to be shown. The first one relies on *Hilbert's Nullstellensatz* whereas the second is quite trivial.
"$\subset$": Let $f \in \mathbf{I}(\mathbf{V}(I))$. That means f vanishes on $\mathbf{V}(I)$ by definition. Using *Hilbert's Nullstellensatz* (Theorem 24) there exists an integer $m \geq 1$ with $f^m \in I$. So $f \in \sqrt{I}$.
"$\supset$": Let $f \in \sqrt{I}$. By definition $f^m \in I$ for some integer $m \geq 1$. So f^m vanishes on $\mathbf{V}(I)$, i.e. $f \in \mathbf{I}(\mathbf{V}(I))$. □

We now arrive at the ideal-variety correspondence. We have seen by *Hilbert's Nullstellensatz* that the important objects are in fact radical ideals and not general ideals. At this stage we see why it is helpful to work over algebraically closed fields. The *Strong Nullstellensatz* holds only for k being algebraically closed. We did not see this explicitly in the proof since it enters the proof of the *Weak Nullstellensatz* from which we concluded Theorem 24 which was used for the *Strong Nullstellensatz*. Over not algebraically closed fields we do not have the ideal-variety correspondence and everything becomes nasty.
The ideal-variety correspondence will be summarized in Table 2.1 on page 18. But before, we have to find the algebraic analogue of the intersection and the union of two varieties. It matches the sum and the product or the intersection of ideals.

Theorem 27. *If I and J are ideals in $k[x_1, \ldots, x_n]$, then $\mathbf{V}(I + J) = \mathbf{V}(I) \cap \mathbf{V}(J)$.*

Proof. "$\subset$": Let $x \in \mathbf{V}(I+J)$. Then $x \in \mathbf{V}(I)$ since $I \subset I+J$ and at the same time $x \in \mathbf{V}(J)$ for the same reason. Therefore $x \in \mathbf{V}(I) \cap \mathbf{V}(J)$.
"$\supset$": Let $x \in \mathbf{V}(I) \cap \mathbf{V}(J)$ and $h \in \mathbf{V}(I + J)$ arbitrary. The there exist $f \in I$ and $g \in J$ with $h = f + g$. We have: $f(x) = 0 = g(x)$. Therefore $h(x) = 0$ as well. Since h was arbitrary x must be in $\mathbf{V}(I + J)$. □

Sums of ideals, i.e. intersection of varieties, will be important for finding the matter curves. Additionally, we would like to decompose our varieties into components. Therefore we must be able to calculate the union of varieties. On the algebraic side the corresponding operation is the product or the intersection of ideals.

Theorem 28. *If I and J are ideals in $k[x_1, \ldots, x_n]$, then $\mathbf{V}(I) \cup \mathbf{V}(J) = \mathbf{V}(I \cap J) = \mathbf{V}(IJ)$.*

Proof. We have to show three implications.

Claim: $\mathbf{V}(I) \cup \mathbf{V}(J) \subset \mathbf{V}(I \cap J)$.

Let $x \in \mathbf{V}(I) \cup \mathbf{V}(J)$. That means that $x \in \mathbf{V}(I)$ or $x \in \mathbf{V}(J)$, i.e. either $f(x) = 0$ for all $f \in I$ or $f(x) = 0$ for all $\in J$. Either way $f(x) = 0$ for all $f \in I \cap J$. Therefore $x \in \mathbf{V}(I \cap J)$. #

Claim: $\mathbf{V}(I \cap J) \subset \mathbf{V}(IJ)$.

We know that $IJ \subset I \cap J$. Therefore $\mathbf{V}(I \cap J) \subset \mathbf{V}(IJ)$. #

Claim: $\mathbf{V}(IJ) \subset \mathbf{V}(I) \cup \mathbf{V}(J)$.

Let $x \in \mathbf{V}(IJ)$. Then $f(x)g(x) = 0$ for all $f \in I$ and all $g \in J$. Now, two cases can occur.

1. $f(x) = 0$ for all $f \in I$. Then $x \in \mathbf{V}(I)$.
2. $f(x) \neq 0$ for some $f \in I$. Then $g(x)$ has to vanish for all $g \in J$.

Either way $x \in \mathbf{V}(I) \cup \mathbf{V}(J)$. #

□

Generally, we only have that $IJ \subset I \cap J$ but in this context both operations lead to the same variety. Therefore we can conclude that $\sqrt{IJ} = \sqrt{I \cap J}$. In algorithmic computations it is easier to compute the intersection of two ideals. This is related to the following proposition.

Proposition 29. *If I, J are ideals, then*

$$\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}.$$

Proof. [CLO10, p. 191] □

2.1.3 The Zariski Topology ([Rei01],[CLO10])

A general variety is a very complicated object. As always it is desirable to decompose it into simpler objects. To do this properly we need the notion of topology on our k^n to define what an irreducible component in a topological sense is. Since we are working with polynomial equations we simply define algebraic sets, i.e. the zero sets of polynomials, to be closed. This is possible since the mapping $\mathbf{V}$ has the following properties:

Proposition 30. *Let Σ be an index set (possibly infinite) and $I, I_\lambda, J \subset k[x_1, \ldots, x_n]$ be ideals for all $\lambda \in \Sigma$. Then:*

1. $\mathbf{V}(\{0\}) = k^n$ *and* $\mathbf{V}(k[x_1, \ldots, x_n]) = \varnothing$,

2. $I \subset J$ *then* $\mathbf{V}(I) \supset \mathbf{V}(J)$,

3. $\mathbf{V}(I \cap J) = \mathbf{V}(I) \cup \mathbf{V}(J)$ *and*

4. $\mathbf{V}(\sum_{\lambda \in \Sigma} I_\lambda) = \bigcap_{\lambda \in \Sigma} \mathbf{V}(I_\lambda)$.

Proof. *1.*, *2.* and the "$\supset$" part of *3.* are obvious.
Ad *3.*: "$\subset$": Show the statement by contraposition. Let $p \notin \mathbf{V}(I_1) \cup \mathbf{V}(I_2)$. So there exist $f \in I_1$ and $g \in I_2$ such that $f(p) \neq 0 \neq g(p)$. By Proposition 14 we know that $fg \in I_1 I_2 \subset I_1 \cap I_2$. But $fg(p) \neq 0$. So $p \notin \mathbf{V}(I_1 \cap I_2)$.
Ad *4.*: The infinite sum is well-defined since $k[x_1, \ldots, x_n]$ is a Noetherian ring.[2] So it suffices to show the statement for finite Σ which is obvious. □

Definition 31. *The* Zariski topology *on k^n is the topology where the closed sets are the algebraic sets of k^n, i.e. all images of $\mathbf{V}$.*

Proposition 30 shows that this in fact is a topology: The empty set and the whole space are closed (*1.*), finite unions of closed sets are closed (*3.*) and infinite intersections of closed sets are closed (*4.*).
It is important to note that the Zariski topology is a relatively weak topology and different from $\mathbb{R}^n$ endued with the standard topology.

Example 32. *The only closed subsets (Zariski topology) of $\mathbb{C}^1$ are whole $\mathbb{C}^1$ and all finite subsets of $\mathbb{C}^1$.*

Proposition 33. *Three properties of the connection between the Zariski topology on $\mathbb{C}^n$ and the standard topology on $\mathbb{C}^n$ are:*

1. *A Zariski closed set of $\mathbb{C}^n$ is closed with respect to the standard topology of $\mathbb{C}^n$ as well.*

2. *Non-empty Zariski open sets are dense in $\mathbb{C}^n$ with respect to the standard topology.*

3. *The Zariski topology is not Hausdorff (as opposed to the standard topology).*

Proof. To *1.* Polynomials are continuous functions with respect to the standard topology and inverse images of closed sets (in this case $\{0\}$) are closed.
To *2.* A non-empty Zariski open set is the complement of a variety.
To *3.* Let U_1, U_2 be non-empty open subsets of $\mathbb{C}^n$. We have to show that $U_1 \cap U_2 \neq \varnothing$. There exist closed sets $V_i = (U_i)^c$ and

$$U_1 \cap U_2 = (V_1)^c \cap (V_2)^c = (V_1 \cup V_2)^c$$

where one of De Morgan's laws was used in the last equality.

[2] Every ascending chain of ideals ($I_1 \subseteq I_2 \subseteq I_3 \subset \cdots$) becomes stationary.

Claim: Let $V_1, V_2 \subsetneq \mathbb{C}^n$ be closed. Then $V_1 \cup V_2 \subsetneq \mathbb{C}^n$.

Suppose $V_1 \cup V_2 = \mathbb{C}^n$. Then $\mathbf{I}(V_1) \cap \mathbf{I}(V_2) = \{0\}$. But the intersection of two ideals is only trivial if one of the ideals is already trivial[a], say $\mathbf{I}(V_1)$, which implies $V_1 = \mathbb{C}^n$. This is a contradiction. #

[a]The greatest common divisor of all generators is element of the intersection.

It is shown by the above equation that the intersection of two arbitrary open sets is non-empty, i.e. the topology is not Hausdorff. □

Additionally, one can show that this nicely fits together with the mappings $\mathbf{I}$ and $\mathbf{V}$.

Proposition 34. *The closure of a given set S with respect to the Zariski topology, i.e. the "smallest" variety containing S, is $\mathbf{V}(\mathbf{I}(S))$.*

Proof. [CLO10, p. 193] □

2.1.4 Decomposition of a Variety into Irreducible Components ([CLO10],[CLO05])

Since we now know a proper topology on our space k^n we can apply the definition of an irreducible topological space.

Definition 35. *A subset $S \subset k[x_1, \ldots, x_n]$ is* irreducible *if it cannot be decomposed into two distinct proper closed subsets.*

Or equivalently in this case: A variety $V \subset k[x_1, \ldots, x_n]$ is *irreducible* if whenever V is written in the form $V = V_1 \cup V_2$ (V_1, V_2 varieties), then either $V_1 = V$ or $V_2 = V$.
This reminds us of prime numbers in $\mathbb{Z}$. And indeed, prime ideals are the precise counterpart to irreducible varieties.

Proposition 36. *Let $V \subset k^n$ be an variety. Then V is irreducible if and only if $\mathbf{I}(V)$ is a prime ideal. For k algebraically closed this sets up a one-to-one correspondence between irreducible varieties in k^n and prime ideals in $k[x_1, \ldots, x_n]$.*

Proof. [Eis04, p. 88] □

With the knowledge that $k[x_1, \ldots, x_n]$ is a Noetherian ring it is not surprising that every variety can be decomposed into the union of a finite number of irreducible varieties.

Theorem 37. *Let $V \subset k^n$ be a variety. Then V can be written as a finite union of irreducible varieties.*

Proof. Let V be not irreducible. Otherwise the statement is trivial. Assume that V cannot be written as a finite union of irreducible varieties. Then there exist varieties V_1 and V_1' such that

- $V = V_1 \cup V_1'$

- $V \neq V_1$ and $V \neq V_1'$
- V_1 and V_1' are themselves not both finite unions of irreducible varieties. Otherwise V would be a finite union as well. Possibly one of the two varieties is a finite union. This shall be V_1'.

Apply the above argument not on V but on V_1 and so forth. This leads to an infinite descending sequence of varieties:

$$V \supsetneq V_1 \supsetneq V_2 \supsetneq \dots$$

Applying the map $\mathbf{I}$ gives an infinite ascending sequence of ideals in $k[x_1, \dots, x_n]$.

$$\mathbf{I}(V) \subsetneq \mathbf{I}(V_1) \subsetneq \mathbf{I}(V_2) \subsetneq \dots$$

$k[x_1, \dots, x_n]$ is a Noetherian ring, i.e. every ascending sequence of ideals has to stabilize. This is a contradiction. □

Example 38. *Let us exemplify the decomposition of a variety into irreducibles. Consider the variety $V = \mathbf{V}(xy) \subset \mathbb{C}^2$. We note that $\langle xy \rangle = \langle x \rangle \cap \langle y \rangle$.*[3] *So $V = \mathbf{V}(xy) = \mathbf{V}(\langle x \rangle \cap \langle y \rangle) = \mathbf{V}(x) \cup \mathbf{V}(y)$.*

The question arises whether or not such a decomposition of varieties is unique. This is possible as long as we consider only minimal decompositions which is again only a technicality.

Definition 39. *Let $V \subset k^n$ be a variety. A decomposition*

$$V = V_1 \cup \dots \cup V_m$$

where each V_i is an irreducible variety is called a minimal decomposition *if $V_i \not\subset V_j$ for $i \neq j$.*

Theorem 40. *Let $V \in k^n$ be a variety. Then V has a minimal decomposition*

$$V = V_1 \cup \dots \cup V_m.$$

This minimal decomposition is unique up to the order in which the V_is are written if k is algebraically closed.

Proof. [CLO10, p. 207] □

As an aside, this gives us the respective statements about radical ideals for free: Every radical ideal not equal to $k[x_1, \dots, x_n]$ can be written as a finite intersection of prime ideals if k is algebraically closed.

[3] Let $f \in \langle x \rangle \cap \langle y \rangle$. That means $f \in \langle x \rangle$ and $f \in \langle y \rangle$. Then $f = c \cdot x = d \cdot y$ for $c, d \in k[x_1, \dots, x_n]$. So $c = \tilde{c}y$ and $d = \tilde{c}x$ for $\tilde{c} \in k[x_1, \dots, x_n]$. Therefore $f = \tilde{c}xy$, i.e. $f \in \langle xy \rangle$. The other direction is proven by Proposition 14.

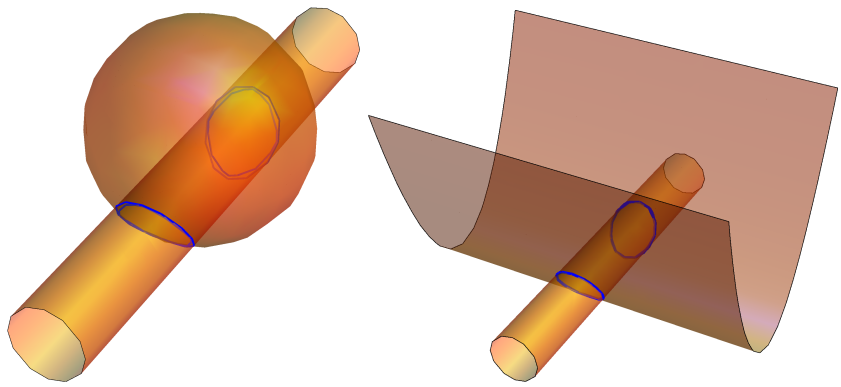
Example 41. *Consider* $f, g \in \mathbb{C}[x, y, z]$ *with* $f = x^2 + y^2 + (z-1)^2 - 4$ *and* $g = x^2 + z^2 - 1$. *Clearly, the real part of* $\mathbf{V}(f)$ *is a 2-sphere centred at* $(0, 0, 1)$ *and the one of* $\mathbf{V}(g)$ *is a cylinder.*
Now, we can take $\mathbf{V}(f) \cup \mathbf{V}(g) = \mathbf{V}(\langle f \rangle \cap \langle g \rangle) = \mathbf{V}(fg)$. *This is the union of two varieties. Additionally, we can intersect the two varieties:*

$$\mathbf{V}(f) \cap \mathbf{V}(g) = \mathbf{V}(\langle f, g \rangle) = \mathbf{V}(\langle x^2 + y^2 + (z-1)^2 - 4, x^2 + z^2 - 1 \rangle).$$

The real part of the intersection is depicted in Figure 2.1 by blue lines. If we compute the intersection using a computer algebra system it could produce the following output

$$\mathbf{V}(f) \cap \mathbf{V}(g) = \mathbf{V}(\langle \tilde{f}, \tilde{g} \rangle) \text{ with } \tilde{f} = 2y^2 - 4z - 5 \text{ and } \tilde{g} = 2x^2 + 2z^2 - 1.$$

This shows that a variety can be possibly described by different polynomials.



(a) Varieties defined by f and g (b) Varieties defined by $\tilde{f}$ and $\tilde{g}$

Figure 2.1: Intersection of two varieties

All important results that have been collected so far are outlined in Table 2.1.

ALGEBRA		**GEOMETRY**
radical ideals		varieties
I	$\rightarrow$	$V(I)$
$I(V)$	$\leftarrow$	V
addition of ideals		intersection of varieties
$I + J$	$\rightarrow$	$V(I) \cap V(J)$
$\sqrt{I(V) + I(W)}$	$\leftarrow$	$V \cap W$
product / intersection of ideals		union of varieties
IJ or $I \cap J$	$\rightarrow$	$V(I) \cup V(J)$
$\sqrt{I(V)I(W)}$ or $I(V) \cap I(W)$	$\leftarrow$	$V \cup W$
prime ideal	$\leftrightarrow$	irreducible variety
ascending chain condition	$\leftrightarrow$	descending chain condition

Table 2.1: Algebra-Geometry Dictionary: 1-to-1 correspondance between ideals and varieties for k algebraically closed

2.1.5 Towards Computations [CLO10]

Now we wish to compute such a minimal decomposition of a general variety. Computer algebra systems (e.g. `Singular`) perform primary (not prime) decompositions. The concept is both similar and a bit more complicated. The ideal is not decomposed into a finite intersection of prime ideals but of primary ideals. This is done because the given ideal is possibly not a radical ideal and the above statement (Theorem 40) cannot be applied.

Definition 42. *An ideal $I \subset k[x_1, \ldots, x_n]$ is* primary *if the following implication holds:*

$$fg \in I \quad \Rightarrow \quad f \in I \text{ or } g^m \in I \text{ for some integer } m \geq 1.$$

Obviously, prime ideals are primary ideals.
To get back to our prime decomposition we realise that every primary ideal defines a unique prime ideal.

Lemma 43. *If I is a primary ideal then $\sqrt{I}$ is prime and the smallest prime ideal containing I.*

This leads to the following definition.

Definition 44. *If I is primary and $\sqrt{I} = P$ then I is a P-*primary ideal.

To get a bit of a feeling what primary ideals are let us consider an example in a less complicated ring.

Example 45. *Consider $R = \mathbb{Z}$. Then there exists the prime factorization of an integer $n \in \mathbb{Z}$ into powers of distinct primes: $n = \pm p_1^{d_1} \cdots p_l^{d_l}$. Therefore we may write the ideal $\langle n \rangle$ as*

$$\langle n \rangle = \langle p_1^{d_1} \rangle \cap \ldots \cap \langle p_l^{d_l} \rangle.$$

This is possible.[4] *The associated primes of this primary decomposition are the $\langle p_i \rangle$.*

Now, we can decompose any ideal including non-radical ones.

Theorem 46. *Every ideal $I \subset k[x_1, \ldots, x_n]$ can be written as a finite intersection of primary ideals.*

Proof. [CLO10, pp. 210–211] □

This can be stated more precisely.

Definition 47. *A* primary decomposition *of an ideal I is an expression of I as an intersection of primary ideals: $I = \bigcap_{i=1}^{r} Q_i$. It is called* minimal *if the $\sqrt{Q_i}$ are all distinct and $Q_i \not\supseteq \bigcap_{j \neq i} Q_j$.*

[4] By induction on l it suffices to show that if $I = \langle p_1^{d_1} \rangle$ and $J = \langle p_1^{d_1} \rangle \cap \ldots \cap \langle p_l^{d_l} \rangle$ then $IJ = I \cap J$. The "$\subset$"-direction is clear (Proposition 14). The converse direction holds since $I + J = R$ (all p_is do not have common divisors). Therefore we can write $1 = x + y$ with $x \in I, y \in J$. Let $f \in I \cap J$. Then $f = 1f = (x + y)f = xf + yf \in IJ + JI = IJ$ which shows the statement.

Theorem 48 (Lasker-Noether)**.** *Every ideal $I \subset k[x_1, \ldots, x_n]$ has a minimal primary decomposition.*

Proof. [CLO10, p. 211] □

Example 49. *This is an example that a primary decomposition does not need to be unique. Consider $I = \langle x^2, xy \rangle$. This ideal can be decomposed into $I = \langle x \rangle \cap \langle x^2, xy, y^2 \rangle$ or into $I = \langle x \rangle \cap \langle x^2, y \rangle$. It is striking that the radicals of $\langle x^2, xy, y^2 \rangle$ and of $\langle x^2, y \rangle$ are the same: $\langle x, y \rangle$. Additionally, note that one component of the primary decomposition is embedded in the other: $\mathbf{V}(x^2, y) \subset \mathbf{V}(x)$.*
Nonetheless, the above condition for a minimal primary decomposition is satisfied:

$$\langle x \rangle \nsubseteq \langle x^2, y \rangle$$
$$\text{and } \langle x^2, y \rangle \nsubseteq \langle x \rangle$$

since $x \notin \langle x^2, y \rangle$ and $y^2 \notin \langle x \rangle$ and likewise for $\langle x^2, xy, y^2 \rangle$.
Finally, $\sqrt{I} = \langle x \rangle$ (proof here[5]) so the prime decomposition of the radical of I coincides with the primary components of I (without embedded components).

The above can happen in general. However, one can show that the radicals of the primary components are independent of the primary decomposition. The associated primes (excluding embedded components) are exactly the same as in the above prime decomposition of the corresponding radical ideal (see [CLO10, p. 211]).
So generally, given any ideal $I \subset k[x_1, \ldots, x_n]$ with k algebraically closed a primary decomposition produces a finite set of irreducible varieties (possibly embedded in each other) and, given a radical ideal, the primary decomposition coincides with the decomposition into irreducibles if embedded components are omitted.
In `Singular` a method for computing the associated primes to the primary ideals is implemented. Note that `Singular` identifies the components which are embedded in another one and gives only the "smallest" ideal ("biggest" variety). In Example 49 `Singular` would return simply $\langle x \rangle$ as associated prime since $\langle x, y \rangle$ the other associated prime fully contains $\langle x \rangle$.
Therefore we can perform the more general primary decomposition and then simply use the associated prime ideals. These will be unique and will be our way to decompose our variety into irreducible ones. Using this method the result will be unique although the intermediate step is not unique.

[5] "$\supset$": Let $f \in \langle x \rangle$, i.e. $f = hx$ for $h \in k[x, y]$. So $f^2 = h^2x^2 \in \langle x^2 \rangle \subset \langle x^2, xy \rangle$.
"$\subset$": Obviously, $(y - a) \notin \sqrt{I}$ for all $a \in k$. So no y can appear as generator of $\sqrt{I}$ and x appears at lowest order. □

2.2 Dimension of a Variety

In the sequel, we will see in which sense varieties are special when compared to manifolds. The crucial difference comes along with different properties of the concept of "dimension". A manifold has a well-defined, i.e. constant along the whole manifold, dimension whereas the dimension of a variety can differ from point to point. Nevertheless, we can define the dimension of a whole variety. This will be the minimum of the dimension at every point. First we will define the notion of dimension in a topological way and then in terms of algebraic objects. We can show that both of them are the same when considering the Zariski topology. Those are global properties.
Afterwards, we define the tangent space at a point first in a geometric and then in an algebraic manner. The points in which the dimension of the respective tangent space and the global dimension coincide will be called regular points and all others singular points. So we are sure that at regular points our (geometric) intuition and the algebraic objects do not differ.
Additionally, we will develop some intuition for the behaviour of the dimension at singular points.

2.2.1 Topological View on Dimension ([Her09],[Per08])

If we think of varieties as topological spaces (endued with the Zariski topology) the term dimension can be defined using irreducible subsets.

Definition 50. *Let X be a non-empty topological space. Then we call*

$$\dim(X) := \sup\{n \in \mathbb{N} \ : \ \exists \text{ irreducible subsets } V_i \text{ with } \varnothing \neq V_0 \subsetneq \ldots \subsetneq V_n \subsetneq X\}$$

the (topological) dimension *of X.*

This already leads us to some insights into the topological dimension which are directly applicable to varieties.

Proposition 51. *If Y is a topological subspace of X then $\dim Y \leq \dim X$. If X is moreover irreducible and of finite dimension and Y is a closed proper subset of X then $\dim Y < \dim X$.*

Proof. [Per08, p. 70] □

Proposition 52. *Let X be a topological space and $X = \bigcup_{i=1}^{n} X_i$, where the sets X_i are closed. Then $\dim X = \sup \dim X_i$.*

Proof. [Per08, p. 70] □

The above proposition shows that the whole problem of computing the dimension of a variety is reduced to the problem of computing the dimension of irreducible varieties since we already know that we can decompose varieties into irreducible ones.

For a better understanding of the topological dimension the following definition shows that this type of dimension is not considered when speaking about $\mathbb{R}^n$ or $\mathbb{C}^n$ endued with the standard topology.

Proposition 53. *Let X be a non-empty topological Hausdorff space. Then* $\dim X = 0$.

Proof. The only closed irreducible subset of a non-empty Hausdorff space is a point. □

Since the Zariski topology is not Hausdorff (Proposition 33) the topological dimension has a chance to be non-trivial.

2.2.2 Algebraic View on Dimension ([Her09],[Per08])

In the following, we shall define the notion of dimension in an algebraic manner. The question is which algebraic object shall be endued with a dimension. To answer this we remember that we have already set up the *ideal-variety correspondence*. Therefore it is desirable to define a dimension of ideals. However, there are good arguments that it is more practical to define the dimension of a ring and set the dimension of an ideal $I \subset R$ to be the dimension of the quotient ring R/I.

Definition 54. *The* Krull dimension *of a ring R is the maximum of the lengths of chains of prime ideals of R. We denote it by* $\dim_K R$.

As already indicated we would like to compute the dimension of e.g. $k[x,y,z]/\langle x^2+y^2+z^2-1\rangle$. This is called the coordinate algebra of the variety defined by $x^2 + y^2 + z^2 - 1 = 0$.
We now define the coordinate algebra of a variety and explore some of the most important facts related to it.

Definition 55. *Let $V \subset k^n$ be a variety. Then we call $k[x_1,\dots,x_n]/\mathbf{I}(V) \equiv \mathcal{O}(V)$ the* coordinate algebra *of V.*

To get some intuition I describe two different ways to think of the coordinate algebra.

- View the elements of the coordinate algebra as functions on the set of points of V: We take the coordinate algebra to be the polynomials in $k[x_1, \dots, x_n]$ and identify two polynomials with each other if they only differ by a polynomial which vanishes on the variety.

- View a point $a \in V$ as a homomorphism $ev_a : \mathcal{O}(V) \to k$ defined by $ev_a(\phi) = [\phi](a)$ where $[\phi](a)$ is the image of $\phi \in k[x_1, \dots, x_n]$ under the quotient map $(k[x_1, \dots, x_n] \to \mathcal{O}(V))$. Obviously, the definition of the evaluation map ev_a does not dependent on the choice of a representative.

When doing algebraic geometry in the abstract way the object of consideration is the coordinate algebra and not the variety itself. Then one has to establish a relation of properties of the variety and properties of the coordinate algebra. Since we were considering the decomposition of varieties I give the corresponding property for the coordinate algebra as an aside. We won't need it in the sequel.

Proposition 56. *Let V be a variety. Then V is irreducible if and only if $\mathcal{O}(V)$ has no zero divisors.*

Proof. "$\Rightarrow$": Let V be irreducible, $a, b \in \mathcal{O}(V)$ with $ab = 0$ and let A, B be their representatives in $k[x_1, \ldots, x_n]$. Then in $\mathcal{O}(V)$ we have $ab = FG + \mathbf{I}(V) = 0$. So FG vanishes on V. Since k is a field and therefore has no zero divisors $FG = 0$ implies $F = 0$ or $G = 0$, i.e. $V \subset \mathbf{V}(F) \cup \mathbf{V}(G)$. So $V = V_1 \cup \mathbf{V}_2$ with $V_1 := V \cap \mathbf{V}(F)$ and $V_2 := V \cap \mathbf{V}(G)$. Since $V, \mathbf{V}(F), \mathbf{V}(G)$ are closed so are V_1 and V_2. Now V is irreducible, so one of the two components has to be equal to V, say V_1. But this implies $V \subset \mathbf{V}(F)$ which means that $F \in \mathbf{I}(V)$ and $a = 0$.
"$\Leftarrow$": Let $\mathcal{O}(V)$ not have any zero divisors. Assume $V = V_1 \cup V_2$ with $V_1 \nsubseteq V_2$ and $V_2 \nsubseteq V_1$, i.e. V reducible. Then there exist polynomials $F \in \mathbf{I}(V_1) \setminus \mathbf{I}(V_2), G \in \mathbf{I}(V_2) \setminus \mathbf{I}(V_1)$. This means $FG \in \mathbf{I}(V_1 \cup V_2)$ and $(F + \mathbf{I}(V))(G + \mathbf{I}(V)) = 0$ in $\mathcal{O}(V)$. Since $\mathcal{O}(V)$ has no zero divisors one of the two factors has to be zero, say the first. But then $F \in \mathbf{I}(V)$ which contradicts to the choice of F. Therefore V has to be irreducible. □

Note that the coordinate algebra of a variety is given by $k[x_1,\ldots,x_n]/\mathbf{I}(\mathbf{V}(I))$ and not by $k[x_1,\ldots,x_n]/I$. Otherwise the above proposition would not hold: $\mathbf{V}(x^2) \subset \mathbb{C}$ is irreducible but $\mathbb{C}[x]/\langle x^2 \rangle$ has a zero divisor namly x. Actually, the coordinate algebra is $\mathbb{C}[x]/x$ where the *Strong Nullstellensatz* was applied.
Important subsets of $\mathcal{O}(V)$ are the sets of functions which vanish at one point. Such a subset is a maximal ideal and needed to define the (algebraic) Zariski tangent space at a given point of a variety (this will be done in Proposition 62).

Definition 57. *Let V be a variety and $x \in V$. Then*

$$\mathfrak{m}_x := \{f \in \mathcal{O}(V) \ : \ f(x) = 0\}$$

is called the maximal ideal of $\mathcal{O}(V)$ at x.

Proposition 58. *The $\{\mathfrak{m}_x \ : \ x \in V\}$ are the only maximal ideals of $\mathcal{O}(V)$ for k algebraically closed.*

Proof. Let $\mathfrak{m} \subset \mathcal{O}(V)$ be a maximal ideal and let $\phi : k[x_1, \ldots, x_n] \to \mathcal{O}(V)$ be the quotient map. Then $\phi^{-1}(\mathfrak{m})$ is maximal. An immediate consequence of the Weak Nullstellensatz (Lemma 23) is that the only maximal ideals of $k[x_1, \ldots, x_n]$ are of the form $\mathfrak{m}_a := \langle x_1 - a_1, \ldots, x_n - a_n \rangle$ for some $(a_1, \ldots, a_n) \in k^n$. So $\phi^{-1}(\mathfrak{m})$ must equal $\mathfrak{m}_a$ for some $a \in k^n$. But that means that the inverse images of elements of $\mathfrak{m}$ vanish at a, i.e. the following implication is true:

$$\bar{f} \equiv \phi(f) \in \mathfrak{m} \subset \mathcal{O}(V) \quad \Rightarrow \quad f(a) = 0.$$

So far we showed $\mathfrak{m} \supset \phi(\mathfrak{m}_a)$. But by definition $\mathfrak{m}_a \supset \mathbf{I}(V)$ so $a \in V$ and $\mathfrak{m} = \phi(\mathfrak{m}_a)$. □

Fortunately, the so far introduced topological dimension of a variety and the Krull dimension of its coordinate algebra fit together nicely.

Proposition 59. *Let V be an irreducible variety and let be $\mathcal{O}(V)$ be the algebra of regular functions on V. Then* $\dim V = \dim_k \mathcal{O}(V)$.

Proof. This follows directly from Proposition 36 (one-to-one correspondence between irreducible varieties and prime ideals). □

2.2.3 Geometric View on Dimension ([Her09],[Per08])

The abstract description of dimension of the above paragraphs is to be linked with our intuitive notion of dimension. So far we have dealt with the term dimension in a global context. Now, we consider dimension from a geometric perspective which is a local concept. The most natural way to define the dimension of manifolds is to define the tangent space at each point which is a vector space and then take the vector space dimension as dimension of the manifold. It can be shown that the local dimension at every point remains constant along the whole manifold.
With respect to varieties the dimension can be defined in a similar way. Nonetheless, the dimension will not be constant any more. A first example is $\mathbf{V}(y) \cup \mathbf{V}(x, z) \subset \mathbb{R}^3$ (shown in Figure 2.2). Intuitively, some part of this ariety is one-dimensional and the other part is two-dimensional. It is not yet clear what happens in the point $(0, 0, 0)$.

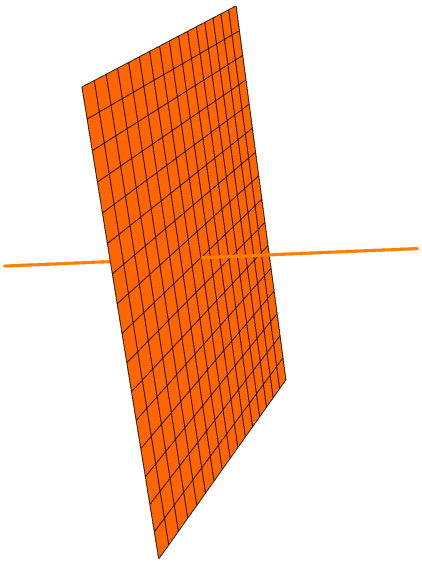


Figure 2.2: A variety ($y = 0$ or $x = z = 0$) with non-constant dimension

Let us begin with the variety analogue of the tangent space in differential geometry.

Definition 60. *Let $V \subset k^n$ be a variety, $p \in V$ a point and $I = \mathbf{I}(V)$.*

- *For $f \in I$ define $f_p^{(1)} := \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big|_p \cdot x_i$.*

- *Let* $I_p := \langle \{ f_p^{(1)} \ : \ f \in I \} \rangle$ *be the ideal generated by all* $f^{(1)}$.
- $T_pV := \mathbf{V}(I_p)$ *is called tangent space at* x. *This is a linear subvectorspace of* k^n.

The dimension of V in p *is defined to be* $\dim_k(T_pV)$ *where* $\dim_k$ *denotes the* k*-vector space dimension.*

Example 61. *Consider the variety* $\mathbf{V}(y^2 - x^3) \subset \mathbb{R}^2$. *Calculate the tangent space at* $p = (0,0)$. *We have*

$$\begin{aligned} I_{(0,0)} &= \left\langle \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)} \cdot x + \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,0)} \cdot y \right\rangle \\ &= \langle 0 \cdot x + 0 \cdot y \rangle = \langle 0 \rangle \end{aligned}$$

The tangent space at $(0,0)$ *is* $\mathbf{V}(\langle 0 \rangle) = k^2$. *It is striking that the dimension of the tangent space at* $(0,0)$ *is 2. Let us compare this to the tangent space at* $(1,1)$.

$$\begin{aligned} I_{(1,1)} &= \left\langle \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(1,1)} \cdot x + \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(1,1)} \cdot y \right\rangle \\ &= \langle -3 \cdot 1 \cdot x + 2 \cdot 1 \cdot y \rangle = \langle -3x + 2y \rangle \end{aligned}$$

So the tangent space at $(1,1)$ *is given by*

$$\begin{aligned} \mathbf{V}(\langle 3x - 2y \rangle) &= \{ (x,y) \in k^2 \ : \ 3x - 2y = 0 \} \\ &= \{ (x, {}^{3x}/_{2}) \in k^2 \} \end{aligned}$$

For better visualization the tangent space in Figure 2.3 is shifted such that it passes through $(1,1)$. *At this point the tangent space has dimension 1.*

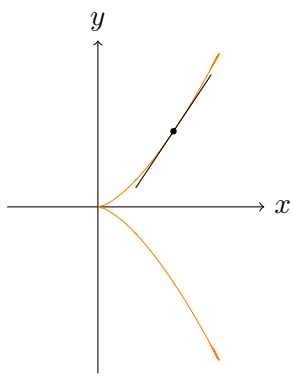


Figure 2.3: Variety defined by $f(x,y) := y^2 - x^3 = 0$

A problem with Definition 60 is that it cannot be applied to an algebraic context since derivatives appear which are intrinsically an analytic tool. The following proposition shows that T_pV can be converted into an algebraic object.

Proposition 62. *Let V be a variety, $x \in V$ and $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}(V)$ corresponding to x. Then there exists a natural isomorphism of vector spaces between T_pV and the dual space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ to $\mathfrak{m}_x/\mathfrak{m}_x^2$. The k-vector space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is called* Zariski tangent space.

Proof. Let $I = \mathbf{I}(V), f \in \mathcal{O}(V)$. Define $d_x f := d_x F$ with $F \in k[x_1, \ldots, x_n]$ some polynomial representative of f. Then the map $d_x f$ is well-defined on the common kernel of the differentials $\{d_x G \ : \ G \in I\}$ which is isomorphic to the tangent space at x (see Proposition 66). Therefore: $d_x f : T_{V,x} \to k$ which is a linear map. This defines the map $d_x : \mathcal{O}(V) \to (T_pV)^*$. Since the differential of a constant function is zero we have $d_x(f - f(x)) = d_x f$. So it suffices to consider $d_x|_{\mathfrak{m}_x}$.

Claim: d_x is onto

First we notice that the differential of any linear map is that linear map.
T_pV is a linear subspace of k^n. Therefore the restriction of a linear map defined on k^n to T_pV gives again a linear map. In this way all linear maps on T_pV can be produced and the restriction to $\mathfrak{m}_x$ is no problem since the differential is not sensitive to overall shifts.
#

Claim: $\ker d_x = \mathfrak{m}_x^2$

"$\supset$": Let $f \in \mathfrak{m}_x^2$. Then there exist $g_1, g_2 \in \mathfrak{m}_x$ with $f = g_1 g_2$. Let $F, G_1, G_2 \in k[x_1, \ldots, x_n]$ be representatives of f, g_1, g_2, i.e. $G_i|_V = g_i$ and $F|_V = f$. Then by the product rule:

$$d_x F = d_x(G_1 G_2) = d_x(G_1) \cdot \underbrace{G_2(x)}_{=0} + \underbrace{G_1(x)}_{=0} \cdot d_x(G_2) = 0.$$

Therefore the derivation of F with respect to all directions is zero including $d_x f = 0$.
"$\subset$": Let $f \in \mathfrak{m}_x$ with $d_x f = 0$ and let $F \in k[x_1, \ldots, x_n]$ be a representative of f. Then $d_x F$ vanishes on T_xV. So there exist a $G \in I$ with $d_x F = d_x G$.
Set $H = F - G$. Since the differential map itself is linear, $d_x H = d_x F - d_x G = 0$. Therefore the Taylor expansion of H has no constant and no linear term which means that $H \in \mathfrak{m}_{k^n,x}^2 := \{f \cdot g \ : \ f, g \in k[x_1, \ldots, x_n], f(x) = g(x) = 0\}$.
Now, H is again a polynomial representative of f (it differs from F only by a polynomial which vanishes on V). Therefore $f \in \mathfrak{m}_x^2$. #

The proposition follows by the *Fundamental theorem on homomorphisms.* □

2.2.4 Singular Points and Singular Locus ([Her09],[Per08], [CLO10])

As mentioned in the physics part of this work the 7-branes may also intersect themselves. It is difficult to define *self-intersection points.* For our purpose it suffices to know that all self-intersection points are so-called *singular points* which are defined and discussed below.

Definition 63. *Let V be an irreducible variety and $p \in V$. p is called* regular point *if* $\dim_k T_pV = \dim V$. *Otherwise p is called* singular.
For reducible varieties we ask for $\dim_k T_pV = \dim_p V$, *where* $\dim_p V$ *is the maximum of the dimensions of all irreducible components passing through p.*

Please note that the notations for the dimension of a k-vector space ($\dim_k$) and the Krull dimension of a ring ($\dim_K$) are very similar.

Proposition 64 (Jacobian criterion)**.** *Let $V \subset k^n$ be an irreducible variety of dimension d. Assume $\mathbf{I}(V) = \langle F_1, \ldots, F_r \rangle$. Then*

$$V \text{ non-singular at } p \Rightarrow \text{rank}\left(\left.\frac{\partial F_i}{\partial x_j}\right|_p\right) = n - d.$$

Proof. [CLO10, p. 492] □

A corollary of the Jacobian criterion is that the local dimension is always equal to or greater the dimension of the variety for irreducible varieties.

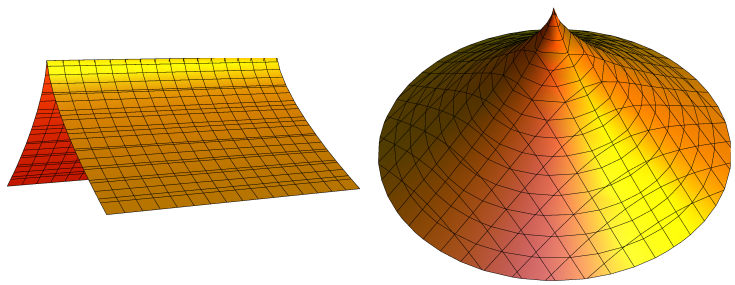
Corollary 65. $\dim T_pV \geq \dim_p V$.

Corollary 66. *The tangent space T_pV is isomorphic to the kernel of the Jacobian matrix:*

$$T_pV \cong \ker\left(\left.\frac{\partial F_i}{\partial x_j}\right|_p\right)_{ij}$$

Let us exemplify this:

Example 67. *Consider the varieties $V = \mathbf{V}(x^3 - y^2)$ and $W = \mathbf{V}(z^3 - x^2 - y^2)$ as subsets of $\mathbb{R}[x, y, z]$ ($n = 3$). Both have dimension $d = 2$.*



(a) $x^3 - y^2 = 0$ (b) $z^3 - x^2 - y^2 = 0$

Figure 2.4: Two examples of singular varieties

The Jacobian matrices are

$$J_V = \left(3x^2, -2y, 0\right)$$
$$\text{and } J_W = \left(-2x, -2y, 3z^2\right).$$

J_V has zero rank iff $x = y = 0$, z arbitrary. J_W has zero rank iff $x = y = z = 0$. Otherwise both matrices have rank 1. Therefore the singular locus of V is $\{(0,0,z) \in \mathbb{R}^3 \ : \ z \in \mathbb{R}\}$ and the singular locus of W is $\{(0,0,0) \in \mathbb{R}^3\}$.
Note that the singular loci are varieties: $\mathbf{V}(x,y)$ and $\mathbf{V}(x,y,z)$.
Computing the tangent space at $p = (0,0,0)$ leads to an important observation. In both cases we obtain $\mathbf{V}(0,0,0) \equiv k^3$ as tangent space although the different shapes of our geometric objects suggest different results since the singular locus of Figure 2.4(a) is 1-dimensional whereas the one of Figure 2.4(b) is zero-dimensional.

Proposition 68. *For a variety V we call*

$$\mathit{Sing}(V) := \{p \in V \ : \ xp \text{ is singular point}\}$$

the singular locus *of V. $\mathit{Sing}(V)$ has the following properties:*

- *$\mathit{Sing}(V)$ is a variety with $\mathit{Sing}(V) \subset V$.*
- *If $p \in \mathit{Sing}(V)$ then $\dim T_pV > \dim V$.*
- *$\mathit{Sing}(V)$ contains no irreducible components of V.*
- *If V_i and V_j are distinct irreducible components then $V_i \cap V_j \subset \mathit{Sing}(V)$.*

Proof. [Her09, p. 42] and [CLO10, p. 490]. □

The last point is well-suited for gaining more intuition about singular points and decomposition of varieties into irreducible ones. Singular points are exactly the points which lie on two irreducible components, i.e. where two irreducible components intersect each other.

Proposition 69. *Let V be a variety and $x \in V$. If there exist irreducible components $V_1 \neq V_2$ of V with $x \in V_1 \cap V_2$, then x is a singular point of V.*

Proof. [Her09, p. 42] □

Therefore *self-intersections* of a variety must be singular points. However, the converse is generally not true (see Example 67).
This discussion about the local dimension of a variety is far from being complete and shall only develop some intuition about the notion of dimension of varieties.

2.3 Computational Aspects ([CLO10])

The above discussion was purely algebraic and did not contain any information about how to implement algorithms to compute e.g. a prime composition of an ideal. The basic object on which all considerations about computational aspects rely is the Gröbner basis of an ideal which will be explored in this section.
The starting question will be: Is every ideal $I \in k[x_1, \ldots, x_n]$ finitely generated, i.e. is there a set of polynomials $\{f_1, \ldots, f_s\}$ such that $I = \langle f_1, \ldots, f_s \rangle$? To answer this question we will have to implement a polynomial division algorithm. In the case $n = 1$, i.e. considering polynomials in only one variable, it is clear that the standard division algorithm is heavily based on the fact that we are able to sort the summands in a proper way. Simply use the order of each summand. For polynomial rings in more than one variable, it is no longer obvious how to sort the summands. For this purpose we need *monomial orderings.*

Definition 70. *A* monomial ordering[6] *on $k[x_1, \ldots, x_n]$ is any relation $<$ on the set of monomials $x^\alpha, \alpha \in \mathbb{Z}^n_{\geq 0}$ satisfying*

1. *$>$ is a total ordering on $\mathbb{Z}^n_{\geq 0}$, i.e. for every $\alpha, \beta \in \mathbb{Z}^n_{\geq 0} : \alpha > \beta$ or $\beta > \alpha$ or $\alpha = \beta$.*
2. *$\alpha > \beta, \gamma \in \mathbb{Z}^n_{\geq 0} \Rightarrow \alpha + \gamma > \beta + \gamma$.*
3. *$>$ is a well-ordering, i.e. every nonempty subset of $\mathbb{Z}^n_{\geq 0}$ has a smallest element under $>$.*

Such a monomial ordering actually exists. The most important examples are the *lexographic* and the *graded reverse lexographic order.*

Definition 71 (Lexographic Order)**.** *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n_{\geq 0}$. We say $\alpha >_{lex} \beta$ if in the vector difference $\alpha - \beta \in \mathbb{Z}^n$ the leftmost nonzero entry is positive.*

Note that there exist $n!$ lexographic orders since the order of the variables needs to be specified. Unless otherwise stated $x_1 > \ldots > x_n$ is assumed.

Definition 72 (Graded Reverse Lex Order)**.** *Let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. We say $\alpha >_{grevlex} \beta$ if*

$$|\alpha| := \sum_{i=1}^{n} \alpha_i > \sum_{i=1}^{n} \beta_i =: |\beta|$$

or

$$|\alpha| = |\beta| \text{ and the rightmost nonzero entry of } \alpha - \beta \in \mathbb{Z}^n \text{ is negative.}$$

This ordering is used in all computations since the performance of computing the Gröbner basis is here the best.

[6] A *monomial* is a polynomial with only one summand.

Example 73. *Let $f = xy^2$ and $g = y^3z^4$. Then $f >_{lex} g$. However, it is in some sense unnatural to say that the degree of f is larger than the one of g. The Graded Reverse Lex Order solves this problem: $f <_{grlex} g$. This is the reason why the graded reverse lex order is defined.*

Before proceeding we need some definitions.

Definition 74. *Let $f = \sum_\alpha a_\alpha x^\alpha \in k[x_1, \ldots, x_n]$ be a nonzero polynomial and let $>$ be a monomial order.*

- *The* multidegree *of f is*

$$multideg(f) = \max(\alpha \in \mathbb{Z}^n_{\geq 0} \ : \ a_\alpha \neq 0).$$

  The maximum is taken with respect to the monomial order.

- *The* leading coefficient *of f is*

$$LC(f) = a_{multideg(f)} \in k.$$

- *The* leading monomial *if f is*

$$LM(f) = x^{multideg(f)}.$$

  Note that this monomial has coefficient 1.

- *The* leading term *of f is*

$$LT(f) = LC(f) \cdot LM(f).$$

Definition 75. *An ideal $I \subset k[x_1, \ldots, x_n]$ is a* monomial ideal *if there is a subset $A \subset \mathbb{Z}^n_{\geq 0}$ such that I consists of all polynomials which are finite sums of the form $\sum_{\alpha \in A} h_\alpha x^\alpha$ where $h_\alpha \in k[x_1, \ldots, x_n]$. In this case, we write $I = \langle x^\alpha \ : \ \alpha \in A \rangle$.*

Of course there are many ideals which are not monomial (e.g. $\langle x+y \rangle \subset \mathbb{C}[x, y]$). Though, they are important since it is relatively easy to show that monomial ideals are finitely generated (see [CLO10, p. 71]). This fact is known as *Dickson's Lemma*. The proof is rather technical and not important in the sequel. So we move on towards the general statement that every ideal in $k[x_1, \ldots, x_n]$ is finitely generated (*Hilbert Basis Theorem*).

Definition 76. *Let $I \subset k[x_1, \ldots, x_n]$ be an non-zero ideal.*

- *$LT(I)$ is the set of leading terms of I, thus*

$$LT(I) = \{cx^\alpha \ : \ \text{there exists } f \in I \text{ with } LT(f) = cx^\alpha\}.$$

- *$\langle LT(I) \rangle$ is the ideal generated by all leading terms of I.*

We notice that for $I = \langle f_1, \ldots, f_s \rangle$ the ideals $\langle LT(f_1), \ldots LT(f_s) \rangle$ and $\langle LT(I) \rangle$ can differ. By definition we know $\langle LT(f_1), \ldots LT(f_s) \rangle \subset \langle LT(I) \rangle$. But equality does not hold in general as we will in Example 80.
However, we can always find some elements in I which lead to equality in the above equations:

Proposition 77. *Let $I \in k[x_1, \ldots, x_n]$ be an ideal. Then:*

- *$\langle LT(I) \rangle$ is an monomial ideal.*
- *There are $g_1, \ldots, g_t \in I$ such that $\langle LT(I) \rangle = \langle LT(g_1), \ldots, LT(g_t) \rangle$.*

Proof. [CLO10, p. 76] □

Theorem 78 (Hilbert Basis Theorem)**.** *Every ideal $I \subset k[x_1, \ldots, x_n]$ has a finite generating set.*

Proof. If $I = \{0\}$ take $\{0\}$ to be the generating set which is finite.
Assume I contains a non-trivial polynomial. By Proposition 77 there exist $g_1, \ldots, g_t \in I$ such that $\langle LT(I) \rangle = \langle LT(g_1), \ldots, LT(g_t) \rangle$.

Claim: $I = \langle g_1, \ldots, g_t \rangle$

"$\supset$": clear.
"$\subset$": Let $f \in I$ and divide f by $g_1, \ldots, g_t$. This gives $f = a_1 g_1 + \ldots + a_t g_t + r$ with no term of r divisible by any of $LT(g_i)$.[a] But then $r = 0$ since $r = f - a_1 g_1 - \ldots - a_t g_t \in I$ and if $r \neq 0$ then $LT(r) \in \langle LT(I) \rangle = \langle LT(g_1), \ldots, LT(g_t) \rangle$ which is a contradiction to the fact that no term of r is divisible by any of $LT(g_i)$. [b]
Thus, $f = a_1 g_1 + \ldots + a_t g_t \in \langle g_1, \ldots, g_t \rangle \quad \Rightarrow \quad I \subset \langle g_1, \ldots, g_t \rangle$. #

[a] The algorithm is a generalization of ordinary polynomial division and can be found e.g. in [CLO10, p. 64]
[b] This is a speciality of monomial ideals. A monomial lies in a monomial ideal if and only if the monomial is divisible by a generator.

In this way we have constructed a finite basis for I. □

Definition 79. *Fix a monomial order. A finite subset $G = \{g_1, \ldots, g_t\}$ of an ideal I is said to be a* Gröbner basis *if*

$$\langle LT(g_1), \ldots, LT(g_t) \rangle = \langle LT(I) \rangle.$$

This means that a generating set of an ideal is a Gröbner basis if every leading term appearing in I is divisible by the leading term of at least one of the g_i.

Example 80. *Consider $I = \langle f_1, f_2 \rangle$ where $f_1 = x^3 - 2xy$ and $f_2 = x^2y - 2y^2 + x$ and use the grlex ordering on monomials in $k[x, y]$. Then $x \cdot f_2 - y \cdot f_1 = x^2$, i.e. $x^2 \in I$ and thus $x^2 \in \langle LT(I) \rangle$. However, x^2 is not divisible by $LT(f_1) = x^3$ or $LT(f_2) = x^2y$ so that $x^2 \notin \langle LT(f_1), LT(f_2) \rangle$. This shows that $\{f_1, f_2\}$ is not a Gröbner basis of I with respect to grlex ordering. A Gröbner basis would be $\langle 2y^2 - x, xy, x^2 \rangle$.* [7]

[7] This has been calculated using `Singular`. Manual computation or a proof would be very tedious.

Gröbner bases have some very good properties when it comes to division algorithms.

Proposition 81. *Let $G = \{g_1, \ldots, g_t\}$ be a Gröbner basis for an ideal $I \subset k[x_1, \ldots, x_n]$ and $f \in k[x_1, \ldots, x_n]$. Then there is a unique $r \in k[x_1, \ldots, x_n]$ with:*

1. *No term of r is divisible by any of $LT(g_i)$.*
2. *There is a $g \in I$ such that $f = g + r$.*

This means that r is the remainder of the division of f by all elements of the Gröbner basis not depending on the ordering of the set G.

Corollary 82. *In the above situation the following holds: $f \in I$ if and only if the remainder on division of f by G is zero.*

This does not go without saying as we will see in the following example.

Example 83. *Consider $I = \langle f_1, f_2 \rangle \subset k[x, y]$ with $f_1 = xy + 1$ and $f_2 = y^2 - 1$. Divide $f = xy^2 - y$ by $\langle f_1, f_2 \rangle$:*

$$xy^2 - x = y \cdot (xy + 1) + 0 \cdot (y^2 - 1) + (-x - y).$$

However, dividing f by $\langle f_2, f_1 \rangle$ yields

$$xy^2 - y = x \cdot (y^2 - 1) + 0 \cdot (xy + 1) + 0.$$

This shows that $f \in I$ but the result of the division has depended on the ordering of the polynomials. Therefore $\{f_1, f_2\}$ cannot be a Gröbner basis because the rest term is not unique.

So, once we have computed the Gröbner basis of an ideal the problem whether or not a given polynomial is element of an ideal is immediately solved. It turns out that Gröbner bases are useful for other issues, too. Examples are simply solving systems of polynomial equations and computing the defining polynomials of a variety given as parametric equations. Many other applications can be found in [Eis04, p. 355].
The important thing for us is that nearly all algorithms which are implemented in computer algebra systems need Gröbner bases as input since polynomial division becomes unique. In the majority of cases ideals are not given in a Gröbner basis. However, there are algorithms to compute the Gröbner basis of an arbitrary ideal. The standard one is the *Buchberger Algorithm.* Since the technicalities on this topic are not essential for physical understanding I will not go into detail here. More on this can be found in [CLO10, pp. 88–94].
The other important function of `Singular` we will use is the computation of the dimension of a variety. This is implemented via the so-called *Hilbert Function.* Much on this can be found in [CLO10, pp. 456–465].

2.4 What Have We Learned?

Let me summarize what we should take into consideration when applying those mathematical tools to F-theory.

- We have the Algebra-Geometry Dictionary when working over an algebraically closed field. Important is:
  - There is a one-to-one correspondence between radical ideals and varieties.
  - There is a one-to-one correspondence between prime ideals and irreducible varieties.
- A variety can be decomposed uniquely into irreducible components.
- The dimension of a variety is the maximum of the dimensions of its irreducible components.
- The points where two or more irreducible components meet are singular points and the dimension in these points is higher than the dimension of the respective irreducible components. However, singular points can arise on irreducible varieties as well.
- The dimension of an irreducible variety equals the dimension at the regular points locally interpreted as a manifold.
- When using a computer algebra system:
  - We have to specify a monomial ordering. As a rule of thumb the grlex ordering will be a very performant one.
  - We have to compute the Gröbner basis of all ideals before processing them. Otherwise correctness of the results is not guaranteed.

3 Computations

In this section the above developed machinery is applied to a given physical problem which arose in [Bor+14]. The input is a set of six polynomial equations. Descriptively, each equation defines an linear approximation to the 7-brane around a specific matter curve. Therefore, always two of them define a matter curve.
Then the intersection locus of different combinations of three of them is computed. As described in the first chapter the elliptic fibration degenerates in those intersection loci. To extract the type of singularity the results of the computation are inserted into the hypersurface equation.
In order to do the computations I will use `Singular` ([Dec+12]) inside a `SAGE` environment ([Ste+14]). A very compact introduction to Singular can be found in [Bac+07].

3.1 Software Set-up

`Singular` can be used stand-alone or as a package in various computer algebra systems like `Maple`, `Mathematica`, `MATLAB` or `SAGE`. The last one is free software and distributed under the terms of GNU GPL. Additionally, I would like to recommend the `sagetex` package for LaTeX. It executes on the fly `SAGE` code written in the `.tex` file and typesets directly the output including plots.

3.2 Implementing the Given Problem

Now we come the the actual computation. Every `Singular` command being used will be explained such that the reader is able to apply the procedure to related problems.

3.2.1 Formulation of the Problem

We will start from three pairs of polynomial equations. Each pair defines (possibly a few) matter curve(s). The procedure how those equations arose can be found in [Bor+14] and is based on an elliptic fibration of B_6. The variables $\{b_i, c_i, d_i\}$ are functions on B_6,

$$0 = d_0 c_2^2 + b_0^2 c_1 - b_0 b_1 c_2, \tag{3.1}$$

$$0 = d_1 b_0 c_2 - b_0^2 b_2 - c_2^2 d_2, \tag{3.2}$$

$$0 = d_0 b_2 c_1 - b_0 b_2^2 - c_1^2 d_2, \tag{3.3}$$
$$0 = d_1 c_1^2 - b_1 b_2 c_1 + b_2^2 c_2, \tag{3.4}$$

$$0 = d_0 c_1^3 c_2^2 + b_0^2 c_1^4 - b_0 b_1 c_1^3 c_2 + c_2^3 (b_1 b_2 c_1 - b_2^2 c_2 - c_1^2 d_1), \tag{3.5}$$
$$0 = d_2 c_1^4 c_2^2 + (b_0 c_1^2 + c_2(-b_1 c_1 + b_2 c_2))(b_0 b_2 c_1^2 + c_2(-b_1 b_2 c_1 + b_2^2 c_2 + c_1^2 d_1)). \tag{3.6}$$

We shall find the intersection loci of codimension three of the matter curves defined by the above equations.

3.2.2 Defining the Polynomial Ring and Ideals

I will now explain how to compute the intersection loci using `Singular`.
First all relevant libraries are loaded. `primdec.lib` provides algorithms for the primary decomposition and `sing.lib` allows to compute the singular locus of a variety.
Then, the ring over which we are working is defined. The characteristic needs to be specified. It is zero in our case. Besides one has to define the variables and the monomial ordering. Then we define the ideals based on the above polynomials.

Important. Before processing any ideal in `Singular` it is necessary to compute its Gröbner basis with the command `.std()`. Otherwise the results may be wrong.

Side remark. When using `Singular` in `SAGE` a "singular." must be added in front of every command. This can be avoided when manipulation already existing objects. Then the standard Python syntax (e.g. `.minAssGTZ()`) can be used like it is done in the after next listing.

```
singular.lib('primdec.lib');
singular.lib('sing.lib');
R = singular.ring(0, '(b0,b1,b2,c1,c2,d0,d1,d2)', 'dp');
I1 = singular.ideal('d0*c2^2+b0^2*c1-b0*b1*c2','d1*b0*c2-b0^2*b2-c2^2*d2').std
    ();
I2 = singular.ideal('d0*b2*c1-b0*b2^2-c1^2*d2','d1*c1^2-b1*b2*c1+b2^2*c2').std
    ();
I3 = singular.ideal('d0*c1^3*c2^2-(-b0^2*c1^4+b0*b1*c1^3*c2+c2^3*(-b1*b2*c1+b2
    ^2*c2+c1^2*d1))','d2*c1^4*c2^2+(b0*c1^2+c2*(-b1*c1+b2*c2))*(b0*b2*c1^2+c2
    *(-b1*b2*c1+b2^2*c2+c1^2*d1))').std();
```

The underlying field

Unfortunately, it is not that simple as it looks. By default Singular computes ideals in $\mathbb{Q}[x_1, \ldots, x_n]$. As having been outlined the ideal-variety correspondence holds only for algebraically closed fields which is not the case here. Therefore it is possible to do the calculations over floating point numbers in $\mathbb{C}$. In order to do this replace the definition of the polynomial ring by the following command.

```
R = singular.ring('(complex,5,I)', '(b0,b1,b2,c1,c2,d0,d1,d2)', 'dp');
```

The five denotes the number of decimal places included in the computation. Unfortunately, this computation does not terminate after a sensible amount of time.
Another possibility is to do the computations over $\mathbb{Q}(i) \equiv \mathbb{Q} + i\mathbb{Q}$. This is implemented via:

```
singular.lib('primitiv.lib');
R1 = singular.ring(0, '(b0,b1,b2,c1,c2,d0,d1,d2,z)', 'dp');
R = singular.splitring('z^2+1');
singular.setring(R);
```

Presumably, the implemented algorithms are based on division algorithms of polynomials. Therefore starting with ideals in $\mathbb{Q}[x_1, \ldots, x_n]$ the results will be in $\mathbb{Q} + i\mathbb{Q}$. Interestingly, the results of the following computation are the same for both $\mathbb{Q}$ and $\mathbb{Q} + i\mathbb{Q}$ as underlying field.

3.2.3 Extracting the Matter Curves

The matter curves possibly consist of different components. We obtain them by computing the minimal associated primes, i.e. decompose the variety into irreducible components.

```
I1pr = I1.minAssGTZ();
I2pr = I2.minAssGTZ();
I3pr = I3.minAssGTZ();
```

The output of `I1pr` is:

```
[1]:
   _[1]=b2^2*d0^2 - b1*b2*d0*d1 + c1*d0*d1^2 + b1^2*b2*d2 - 2*b2*c1*d0*d2
- b1*c1*d1*d2 + c1^2*d2^2
   _[2]=b0*b2*d0 - c2*d0*d1 - b0*c1*d2 + b1*c2*d2
   _[3]=-b2*c2*d0^2 - b0*c1*d0*d1 + b1*c2*d0*d1 + b0*b1*c1*d2 - b1^2*c2*d2
+ c1*c2*d0*d2
   _[4]=b0*b1*b2 - b2*c2*d0 - b0*c1*d1 + c1*c2*d2
   _[5]=b0^2*c1 - b0*b1*c2 + c2^2*d0
   _[6]=b0^2*b2 - b0*c2*d1 + c2^2*d2
[2]:
   _[1]=c2
   _[2]=b0
```

So the curve corresponding to the first two equations has two components:

$$\begin{gathered}\langle c_2, b_0\rangle,\\ \langle b_2^2 d_0^2 - b_1 b_2 d_0 d_1 + c_1 d_0 d_1^2 + b_1^2 b_2 d_2 - 2 b_2 c_1 d_0 d_2 - b_1 c_1 d_1 d_2 + c_1^2 d_2^2,\\ b_0 b_2 d_0 - c_2 d_0 d_1 - b_0 c_1 d_2 + b_1 c_2 d_2, \ldots\rangle.\end{gathered}$$

Next, these components are saved in variables. The notation is the same as in [Bor+14, p. 15].

```
CI1 = I1pr[2].std(); # c2 = 0 = b0
CI2 = I1pr[1].std(); # complicated

CI3 = I2pr[2].std(); # b2 = 0 = c1
CI4 = I2pr[1].std(); # complicated

CI5 = I3pr[2].std(); # c2 = 0 = c1
CI6 = I3pr[1].std(); # complicated
```

As already been indicated we expect the torus fiber to degenerate at the matter curves. This can be seen by a factorization of the hypersurface equation (see [Bor+14, p. 9]),

$$\begin{aligned} P_{T^2} = vw(c_1 w s_1 + c_2 v s_0) + u(b_0 v_2 s_0^2 + b_1 v w s_0 s_1 + b_2 w_2 s_1^2) + \\ + u_2(d_0 v s_0^2 s_1 + d_1 w s_0 s_1^2 + d_2 u s_0^2 s_1^2) = 0, \end{aligned}$$

when plugging in the above curves. For C_1, C_3 and C_5 this can be easily checked, e.g.

$$\begin{aligned} P_{T^2}|_{\mathbf{1}^{(1)}} &\equiv P_{T^2}|_{b_0=0,c_2=0} \\ &= s_1(d_2 s_0^2 s_1 u^3 + d_0 s_0^2 u^2 v + d_1 s_0 s_1 u^2 w + b_1 s_0 u v w + b_2 s_1 u w^2 + c_1 v w^2). \end{aligned}$$

The elliptic fibration allows for two gauge groups: $U(1)_1$ and $U(1)_2$. It is possible to compute the charges of the respective matter curves by analysing the the hypersurface equation. In our case we will observe:

state	locus	$(U(1)_1, U(1)_2)$-charges
$\mathbf{1}^{(1)}/\bar{\mathbf{1}}^{(1)}$	C_1	$(1,-1)/(-1,1)$
$\mathbf{1}^{(2)}/\bar{\mathbf{1}}^{(2)}$	C_2	$(1,0)/(-1,0)$
$\mathbf{1}^{(3)}/\bar{\mathbf{1}}^{(3)}$	C_3	$(1,2)/(-1,-2)$
$\mathbf{1}^{(4)}/\bar{\mathbf{1}}^{(4)}$	C_4	$(1,1)/(-1,-1)$
$\mathbf{1}^{(5)}/\bar{\mathbf{1}}^{(5)}$	C_5	$(0,2)/(0,-2)$
$\mathbf{1}^{(6)}/\bar{\mathbf{1}}^{(6)}$	C_6	$(0,1)/(0,-1)$

Figure 3.1: $(U(1)_1, U(1)_2)$-charges of the matter curves ([LW14, p. 6])

3.2.4 Intersecting the Matter Curves

To find all possible Yukawa interactions we have to respect charge conservation. One possible choice would be $\mathbf{1}^{(1)}\bar{\mathbf{1}}^{(4)}\mathbf{1}^{(5)}$ since the $U(1)_1$-charges add up to $1+(-1)+0=0$ and the $U(1)_2$-charges add up to $-1+(-1)+2=0$. Therefore charge conservation is guaranteed in this case.

All possible combinations are: $\mathbf{1}^{(1)}\bar{\mathbf{1}}^{(4)}\mathbf{1}^{(5)}$, $\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(3)}\mathbf{1}^{(5)}$, $\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(4)}\mathbf{1}^{(6)}$, $\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}$, $\bar{\mathbf{1}}^{(3)}\mathbf{1}^{(4)}\mathbf{1}^{(6)}$ and $\bar{\mathbf{1}}^{(5)}\mathbf{1}^{(6)}\mathbf{1}^{(6)}$. To find the corresponding loci on our branes we intersect the corresponding matter curves (e.g. $C_1 \cap C_4 \cap C_5$). This can be done by adding the ideals.
One of the possible interactions is $C_5 \cap C_6 \cap C_6$. It is not clear how to express $C_6 \cap C_6$ mathematically. Descriptively, we are looking for self intersections of C_6 and self intersections are obviously singular points of the variety. But not every singular point is a self intersection.

```
inters1 = (CI1+CI4+CI5).std();
inters2 = (CI2+CI3+CI5).std();
inters3 = (CI2+CI4+CI6).std();
inters4 = (CI1+CI2+CI6).std();
inters5 = (CI3+CI4+CI6).std();
singCI6 = singular.slocus(CI6).std();
inters6 = (CI5+singCI6).std();
inters61 = (CI5+CI6).std();
```

As already been indicated the variables $\{b_i, c_i, d_i\}$ are themselves polynomial functions on B_6 which is a complex three-dimensional space. One way to imagine what happens is that the variables $\{b_i, c_i, d_i\}$ are related with five scaling relations. In the generic case, we then end up with again three dimensions. If we look for points on B_6, i.e. zero-dimensional subspaces, we have to take into account the five-dimensional components.
To avoid confusion I suggest to count codimensions. We are looking for points on B_6 (codimension 3). This corresponds to a five-dimensional locus in $\mathbb{C}^8$ (again codimension $8-5=3$).

```
inters1pr = inters1.minAssGTZ();
inters2pr = inters2.minAssGTZ();
inters3pr = inters3.minAssGTZ();
inters4pr = inters4.minAssGTZ();
inters5pr = inters5.minAssGTZ();
inters6pr = inters6.minAssGTZ();
inters61pr = inters61.minAssGTZ();

inters1pr[1].std().dim()  # -> codim 3
inters2pr[1].std().dim()  # -> codim 3
inters3pr[1].std().dim()  # -> codim 3 # here .std() important
inters3pr[2].std().dim()  # -> codim 4
inters4pr[1].std().dim()  # -> codim 3
inters5pr[1].std().dim()  # -> codim 3
inters6pr[1].std().dim()  # -> codim 3
inters61pr[1].std().dim() # -> codim 3
inters61pr[2].std().dim() # -> codim 4
```

So we find that $\mathbf{1}^{(1)}\bar{\mathbf{1}}^{(4)}\mathbf{1}^{(5)}, \mathbf{1}^{(2)}\bar{\mathbf{1}}^{(3)}\mathbf{1}^{(5)}, \mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}, \bar{\mathbf{1}}^{(3)}\mathbf{1}^{(4)}\mathbf{1}^{(6)}$ and $\bar{\mathbf{1}}^{(5)}\mathbf{1}^{(6)}$ have only one component and this one has the right codimension. $\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(4)}\mathbf{1}^{(6)}$ and $\bar{\mathbf{1}}^{(5)}\mathbf{1}^{(6)}\mathbf{1}^{(6)}$ are built out of two components where one has the right and the other one the wrong codimension. Additionally, $\bar{\mathbf{1}}^{(5)}\mathbf{1}^{(6)}$ and the first component of $\bar{\mathbf{1}}^{(5)}\mathbf{1}^{(6)}\mathbf{1}^{(6)}$ are exactly the same. This can be interpreted in the following fashion: The curve C_5 intersects the C_6 precisely in some subset of its singular locus.

$\mathbf{1}^{(1)}\bar{\mathbf{1}}^{(4)}\mathbf{1}^{(5)}$	$\langle c_2, c_1, b_0\rangle$
$\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(3)}\mathbf{1}^{(5)}$	$\langle c_2, c_1, b_2\rangle$
$\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(4)}\mathbf{1}^{(6)}$	too long to be displayed
$\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}$	$\langle b_2^2d_0^2 - b_1b_2d_0d_1 + c_1d_0d_1^2 + b_1^2b_2d_2 - 2b_2c_1d_0d_2 - b_1c_1d_1d_2 + c_1^2d_2^2, c_2, b_0\rangle$
$\bar{\mathbf{1}}^{(3)}\mathbf{1}^{(4)}\mathbf{1}^{(6)}$	$\langle -b_0b_1d_0d_1 + c_2d_0^2d_1 + b0^2d_1^2 + b_0b_1^2d_2 - b_1c_2d_0d_2 - 2b_0c_2d_1d_2 + c_2^2d_2^2, c_1, b_2\rangle$
$\bar{\mathbf{1}}^{(5)}\mathbf{1}^{(6)}\mathbf{1}^{(6)}$	$\langle -b_2d_0^2 + b_1d_0d_1 - b_0d_1^2 - b_1^2d_2 + 4b_0b_2d_2, c_2, c_1\rangle$

3.3 Further Analysis

For later analysis we have to plug the formulae in, the following hypersurface equation (see [Bor+14, p. 9]):

$$\begin{aligned} P_{T^2} = vw(c_1ws_1 + c_2vs_0) + u(b_0v_2s_0^2 + b_1vws_0s_1 + b_2w_2s_1^2) + \\ + u_2(d_0vs_0^2s_1 + d_1ws_0s_1^2 + d_2us_0^2s_1^2) = 0 \end{aligned}$$

I will only do this for $\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}$, $\bar{\mathbf{1}}^{(3)}\mathbf{1}^{(4)}\mathbf{1}^{(6)}$ and $\bar{\mathbf{1}}^{(5)}\mathbf{1}^{(6)}\mathbf{1}^{(6)}$ because the other ones could already be analysed by different methods in [Bor+14]. Besides, at least for $\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(4)}\mathbf{1}^{(6)}$ the computation is far too complicated in this manner.

- For $\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}$:
  - Solve the last equation for d_1: $d_1 = \frac{b_1b_2d_0 + b_1c_1d_2 \pm \sqrt{b_1^2 - 4c_1d_0}(b_2d_0 - c_1d_2)}{2c_1d_0}$.
  - Plugging this and $b_0 = 0$ and $c_2 = 0$ in the hypersurface equation it gives:

$$\begin{aligned} P_{T^2}|_{\mathbf{1}^{(1)}\mathbf{1}^{(2)}\bar{\mathbf{1}}^{(6)}} =& \frac{1}{4c_1d_0}s_1\left[\left(b_1 \pm \sqrt{b_1^2 - 4c_1d_0}\right)s_0u + 2c_1w\right] \\ &\times \left[\left(b_1 \mp \sqrt{b_1^2 - 4c_1d_0}\right)d_2s_0s_1u^2 + 2b_2d_0s_1uw + \dots\right. \\ &\left.\dots + \left(b_1d_0 \mp d_0\sqrt{b_1^2 - 4c_1d_0}\right)s_0uv + 2c_1d_0vw\right] \end{aligned}$$

- For $\bar{\mathbf{1}}^{(3)}\mathbf{1}^{(4)}\mathbf{1}^{(6)}$:
  - Solve the last equation for d_2: $d_2 = \frac{-b_0b_1^2 + b_1c_2d_0 + 2b_0c_2d_1 \pm (b_0b_1 - c_2d_0)\sqrt{b_1^2 - 4c_2d_1}}{2c_2^2}$.
  - Plugging this and $b_2 = 0$ and $c_1 = 0$ in the hypersurface equation it gives:

$$\begin{aligned} P_{T^2}|_{\bar{\mathbf{1}}^{(3)}\mathbf{1}^{(4)}\mathbf{1}^{(6)}} =& \frac{1}{4c_2^2}s_0\left[\left(b_1 + \sqrt{b_1^2 - 4c_2d_1}\right)s_1u + 2c_2v\right] \\ &\times \left[\left(-b_0b_1 + 2c_2d_0 \pm b_0\sqrt{b_1^2 - 4c_2d_1}\right)s_0s_1u^2 + \dots\right. \\ &\left.\dots + 2b_0c_2s_0uv + \left(b_1c_2 \pm c_2\sqrt{b_1^2 - 4c_2d_1}\right)s_1uw + 2c_2^2vw\right] \end{aligned}$$

- For $\bar{\mathbf{1}}^{(5)}\mathbf{1}^{(6)}\mathbf{1}^{(6)}$:
  - Solve the last equation for b_1: $b_1 = \frac{d_0 d_1 \pm \sqrt{d_0^2 - 4b_0 d_2}\sqrt{d_1^2 - 4b_2 d_2}}{2d_2}$
  - Plugging this and $c_1 = 0$ and $c_2 = 0$ in the hypersurface equation it gives:

$$
\begin{aligned}
&P_{T^2}|_{\bar{\mathbf{1}}^{(5)}\mathbf{1}^{(6)}\mathbf{1}^{(6)}} = \frac{1}{4d_2}u \\
&\times \left[2d_2 s_0 s_1 u + \left(d_0 - \sqrt{d_0^2 - 4b_0 d_2}\right) s_0 v + \left(d_1 \pm \sqrt{d_1^2 - 4b_2 d_2}\right) s_1 w\right] \\
&\times \left[2d_2 s_0 s_1 u + \left(d_0 + \sqrt{d_0^2 - 4b_0 d_2}\right) s_0 v + \left(d_1 \mp \sqrt{d_1^2 - 4b_2 d_2}\right) s_1 w\right]
\end{aligned}
$$

3.4 What have we seen?

We see that the hypersurface equation factors three times in the above cases. This is expected by construction: The internal Calabi-Yau manifold is endowed with an elliptic fibration. This means that a generic point of the base B_6 is mapped to a torus such that the torus is deformed smoothly when varying the base point continuously. Over specific loci on the base space B_6 the torus fiber degenerates. Descriptively, it is "squeezed" at one point (see Figure 3.2). Then it has the same form as a $\mathbb{P}^1$.

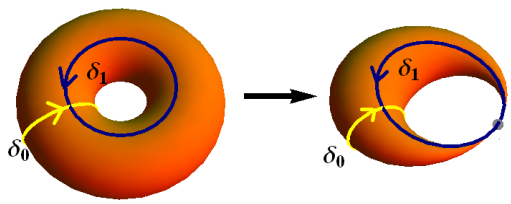


Figure 3.2: Degeneration of a torus into one $\mathbb{P}^1$ (graphics by [Gul+13])

If this happens two times (i.e. a two-times "squeezed" torus or two $\mathbb{P}^1$s) this corresponds to the factorization of the hypersurface equation into two parts. Part of the theory is that these are exactly the matter curves (see page 39).
Additionally, the theory says that over the intersection points of the matter curves we will observe another enhancement meaning that the hypersurface equation factors into three parts which corresponds to a torus which is at three points "squeezed", i.e. three $\mathbb{P}^1$s. This is exactly what we observed in Section 3.3.
Although it is theoretically clear by construction that this three-fold enhancement happens it is a valid cross-check for the correctness of the previous computations.

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Erklärung

Ich versichere, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.
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